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HEREDITARILY PYTHAGOREAN FIELDS, INFINITE HARRISON-PRIMES AND SUMS OF 2^nth POWERS

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In [3] W. D. Geyer studied (infinite) algebraic number fields having an absolute Galoisgroup which is solvable as an abstract group. In particular he showed that for a real number field K of this type the absolute Galoisgroup $G(\bar{K}|K(i))$ must be abelian (we denote the algebraic closure of a field k by $\bar{k}$). Geyer's work may therefore be considered as a generalization of the well-known characterization of real-closed fields given by E. Artin and O. Schreier. This note reports on the work [1] originated in an attempt to carry over Geyer's results to arbitrary formally real fields K (= real fields). We investigate those fields with abelian Galoisgroup $G(\bar{K}|K(i))$ which may be regarded as substitutes for real-closed fields. The orderings of real-closed fields are to be replaced by certain infinite Harrison-primes, and the study of sums of squares by orderings can be extended with help of these Harrison-primes to sums of 2^nth powers.

1. Hereditarily pythagorean fields. A real field K is called *pythagorean* if $K^2 + K^2 = K^2$ holds, *hereditarily pythagorean* (= h.p.) if any real algebraic extension is pythagorean. Let $\mathbf{Z}_p$ be the compact additive group of the p-adic integers, δ_{ij} the Kronecker-symbol and Br(K) the Brauergroup of the real field K.

THEOREM 1. *K is a h.p. field iff $G(\bar{K}|K(i))$ is abelian. If K is h.p., then*

(i) $G(\bar{K}|K) = \langle\sigma\rangle \times G(\bar{K}|K(i))$, $\sigma^2 = 1$, *σ operates by inversion on* $G(\bar{K}|K(i))$,

(ii) $G(\bar{K}|K(i)) \cong \Pi_p\, \mathbf{Z}_p^{\alpha_p}$ *with* $\alpha_p = -\delta_{2p} + \dim_{\mathbf{F}_p} K^\times/K^{\times p}$,

(iii) Br(K) *has exponent* 2, $\dim_{\mathbf{F}_2} \mathrm{Br}(K) = \alpha_2 + \binom{\alpha_2}{2} + 1$

H.p. fields can further be characterized by the Haar-measure of the set of involutions in $G(\bar{K}|K)$ [1], by the existence of a certain henselian valuation [2] (both due to L. Bröcker), by the existence of a Kummer-theory for all algebraic extensions [1] (F. Halter-Koch) or by torsion properties of the Wittring of $K(X)$ [1].

2. Infinite Harrison-primes. An infinite Harrison-prime P [4] of K is called an *ordering of type $n \in \mathbf{N}$* if $K^{2^n} \subset P$ and of *exact type n* if $K^{2^n} \subset P$, $K^{2^{n-1}} \not\subset P$. Orderings of type 1 are the usual orderings. Let Q_n be the subset of all sums of 2^nth powers in K. Then $Q_n = \bigcap P$ where P ranges over all orderings of type n, the case $n = 1$ is due to E. Artin.

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