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HOMOTOPY RIGIDITY OF LINEAR ACTIONS: CHARACTERS TELL ALL

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Our aim is to present a striking rigidity phenomenom in unitary representations of compact groups. Let $U = U(n)$ be a unitary group and H a closed subgroup of U. The homogeneous space U/H is a smooth manifold with a smooth action λ of U induced by left multiplication. If $\alpha: G \to U$ is a representation of the compact group G, then $\lambda \circ (\alpha \times 1): G \times U/H \to U/H$ is an action of G on U/H, and we denote this G-structure by $(U/H, \alpha)$. Such actions of G on U/H are called *linear* actions. We shall give a complete description of the G-homotopy types of linear actions on U/H for a certain class of H. To motivate our results we shall first examine some obvious G-equivalences of linear actions.

If X is a U-space, then the set of U-maps $\mathrm{Map}_U(U/H, X)$ is in one-to-one correspondence with elements $x \in X$ such that $U_x \supset H$, where $U_x = \{u \in U | ux = x\}$ is the isotropy group of the action at x. For example, if $a \in U$ then the element aH in U/H has isotropy group aHa^{-1} and the U-map f: $U/aHa^{-1} \to U/H$ given by $f(uaHa^{-1}) = uaH$ is a U-equivalence. Indeed if H and K are closed subgroups of U then U/H and U/K are U-equivalent if and only if $K = aHa^{-1}$ for a suitable $a \in U$. Suppose α, γ: $G \to U$ are representations such that there exists an $a \in U$ such that $\gamma(g) = a\alpha(g)a^{-1}$ for all $g \in G$ (we say that γ is *similar* to α). The map k: $(U/H, \alpha) \to (U/H, \gamma)$ given by $k(uH) = auH$ is a G-equivalence. Indeed, k is the composition of the G-equivalence $(U/H, \alpha) \to (U/aHa^{-1}, \gamma)$ induced by conjugation with a in U and the U-equivalence (hence G-equivalence!) f: $(U/aHa^{-1}, \gamma) \to (U/H, \gamma)$. Thus similarity of representations gives us G-equivalences of the associated linear actions on U/H. Here is another obvious way of obtaining G-equivalences: let c: $U \to U$ be conjugation by unitary matrices $c(a) = \bar{a}$; then if $c(H) = H$, we obtain a G-equivalence c: $(U/H, \alpha) \to (U/H, \bar{\alpha})$ where $\bar{\alpha} = c \circ \alpha$ is the representation conjugate to α.

It is too much to hope that $(U/H, \alpha)$ is G-homotopy equivalent to $(U/H, \beta)$ if and only if β or $\bar{\beta}$ is similar to α. For example, if H is a subgroup of maximal rank in U and C is the center of U then $C \subset H$ and C acts trivially on U/H, so if we let $P(U) = U/C$ be the projective unitary group (with q: $U \to P(U)$ the quotient map), then the standard left action λ of U on U/H induces an action of $P(U)$ on U/H, and it is the similarity class of the projective representation $q \circ \alpha$: $G \to P(U)$ which matters. We have: if α, β: $G \to U$ are representations and χ: $G \to S^1 = C$ is a homomorphism such that β or $\bar{\beta}$ is similar to $\chi\alpha$ then $(U/H, \alpha)$ is G-equiva-

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