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DERIVATIVES OF POLYNOMIALS OF BEST APPROXIMATION

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A large number of results in the theory of approximation deal with the subject of best approximation, but only a few study the behavior of derivatives of the function of best approximation. Roulier [4] proved recently, that if $f \in C^r[-1, 1]$ and if p_n is the polynomial of degree n of best approximation to f on $[-1, 1]$ in the supremum norm, then if $k \leqslant r/2$,

$$\lim_{n\to\infty} \|f^{(k)} - p_n^{(k)}\|_\infty = 0.$$

Our main result is an estimate for $\|f^{(k)} - p_n^{(k)}\|_\infty$ from below. This estimate can be used to solve certain problems in monotone approximation. The result is obtained by studying the distribution of points of maximum deviation in best approximation.

THEOREM 1. *Let f be a continuous function in $[-1, 1]$, which is not a polynomial. For each $n = 1, 2, \ldots$, let $p_n(x)$ be the polynomial of best approximation to f in the supremum norm, by polynomials of degree n at most and let $-1 \leqslant x_{0,n} < x_{1,n} < \cdots < x_{n+1,n} \leqslant 1$ be a Chebychev alternation for f, i.e., a set of points of maximum deviation at which the signs alternate. Then*

$$\varliminf_{n\to\infty} (1 + x_{k,n})(n/\ln n)^2 \leqslant C(k),$$

$$\varliminf_{n\to\infty} (1 - x_{n+1-k,n})(n/\ln n)^2 \leqslant C(k),$$

where $C(k)$ is a constant depending on k and f only.

The proof of the theorem follows by analyzing the proof of Lemma 2 in [1], modifying the proof of the theorem in [1] and making the necessary substitution. This theorem shows that the points of maximum deviation in the approximation of continuous functions by polynomials are in general, denser close to the endpoints.

THEOREM 2. *Under the same conditions as Theorem 1, if f has k continuous derivatives in $[-1, 1]$, then there is an infinite sequence of natural numbers n_j, such that*

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