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AN INTERPRETATION OF STOCHASTIC DIFFERENTIAL EQUATIONS AS ORDINARY DIFFERENTIAL EQUATIONS WHICH DEPEND ON THE SAMPLE POINT

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We consider: (I) a "deterministic" ordinary differential equation

$$dx(t) = f(x(t))\,dt + g(x(t))\,dw(t), \qquad x \in \mathbf{R}^n, \tag{det}$$

where f, g are vector fields in $\mathbf{R}^n$ and w: $[a, b] \longrightarrow \mathbf{R}$ is a continuous function, and (II) the stochastic equation

$$dX = f(X)\,dt + g(X)\,dW \tag{stoch}$$

where f, g are as before, and $W = \{W_t\colon t \in [a, b]\}$ is a stochastic process with continuous sample paths, defined on a probability space $\mathcal{P} = (\Omega, \mathcal{A}, P)$.

In order to define what is meant by a solution of (stoch), the most obvious approach would go as follows: a process $\{X_t\colon t \in [a, b]\}$ is said to be a solution of (stoch) if, for each sample point $\omega \in \Omega$, the function $t \longrightarrow X_t(\omega)$ is a solution of

$$dx(t) = f(x(t))\,dt + g(x(t))\,dw_\omega(t), \tag{det_ω}$$

where $w_\omega(t) = W_t(\omega)$. Usually, this approach is not followed because of the technical difficulties that appear when one tries to solve (det) for arbitrary continuous w. One is therefore forced to use other lines of attack, and to study (stoch) directly. This gives rise to at least two nonequivalent theories, namely, the one due to Ito, and that of Fisk and Stratonovich.

The purpose of this note is to announce that the "obvious approach" described in the beginning of the preceding paragraph can actually be pursued all the way, leading to a simple construction of solutions of (stoch) for arbitrary processes W with continuous sample paths. When W is a Wiener process, our construction gives the same result as the ordinary solution in the sense of Stratonovich.

First, let us define what is meant by a solution of (det) when w is only continuous. We say that a curve x: $t \longrightarrow x(t)$, $a \leqslant t \leqslant b$, is a *solution* of (det), if there exists a neighborhood U of w in the space $C^0([a, b], \mathbf{R})$ of continuous real-valued functions on $[a, b]$ (with the sup norm), and a continuous map Γ: $U \longrightarrow C^0([a, b], \mathbf{R}^n)$, such that:

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