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THE KUNZE-STEIN PHENOMENON

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ABSTRACT. We show that, if G is a connected semisimple Lie group with finite center, then $L^p(G) * L^2(G) \subseteq L^2(G)$ if $1 \leqslant p < 2$.

The theorem announced in the abstract was proved by R. A. Kunze and E. M. Stein [4] for the case when G is $SL(2, \mathbf{R})$. Various authors have extended to various other groups G, but the methods used have been quite technical and hard to generalise. The difficulty is the expression of the analytic continuation of the principal series as uniformly bounded representations on a Hilbert space. We avoid this difficulty by treating isometric representations on mixed $L^{\mathbf{p}}$-spaces. Here is an outline of our method. We suppose that G has real rank r.

Let $\overline{N}MAN$ be a Bruhat decomposition of G, and $\alpha_1, \ldots, \alpha_r$ the associated simple positive roots. We denote by $\overline{N}_j$ the subgroup of $\overline{N}$ whose Lie algebra is the sum of the root spaces $\mathfrak{N}_{-\alpha}$, where

$$\alpha = m_j\alpha_j + m_{j-1}\alpha_{j-1} + \cdots + m_1\alpha_1,$$

with $m_j > 0$. By ρ_j we denote the element of the dual of the Lie algebra of A, defined by the rule

$$\rho_j(a) = -\tfrac{1}{2}\,\mathrm{tr}[\mathrm{ad}(a)|_{\overline{\mathfrak{N}}_j}],$$

and by ρ the sum of the ρ_j. The group $\overline{N}$ has a decomposition $\overline{N} = \overline{N}_r \cdots \overline{N}_1$. Almost all elements of G have a Bruhat decomposition: we write

$$g = \overline{N}(g)M(g)A(g)N(g).$$

The unitary class-one principal series can be realised on $L^2(\overline{N})$ by the formula

$$[\pi_z(g)\xi](\overline{n}) = \exp[-(\rho + z_1\rho_1 + \cdots + z_r\rho_r)\log A(g^{-1}n)]\,\xi(\overline{N}(g^{-1}n)),$$

where z is a purely imaginary r-tuple. Allowing z to be complex in the above formula, we obtain one "analytic continuation of the principal series" (see Stein [5]). Let $L^{\mathbf{p}}(\overline{N})$ be the space of functions on $\overline{N}$ such that the norm $\|\ \|_{\mathbf{p}}$:

$$\|\xi\|_{\mathbf{p}} = \left[\int_{N_r} dn_r \cdots \left[\int_{N_1} dn_1 |\xi(n_r \cdots n_1)|^{p_1}\right]^{p_2/p_1} \cdots\right]^{1/p_r},$$

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