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A REMARK ON EXPONENTIAL SUMS

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ABSTRACT. The L^1 norm $\|F\|_1$ and the absolute value M of the minimum of the real part of $F(x) = \exp(in_1x) + \cdots + \exp(in_Nx)$, $n_1, \ldots, n_N$ distinct positive integers, satisfy the inequality $M \log M + \|F\|_1 \geqslant \text{Const} \log N$.

Let $0 < n_1 < \cdots < n_N$ be N distinct integers, $c_1, \ldots, c_N$ complex numbers, and write

$$F(x) = f(x) + ig(x) = c_1 \exp(in_1x) + \cdots + c_N \exp(in_Nx).$$

Throughout this note C will denote a positive absolute constant (not always the same), integrals without limits of integration will be understood as taken over $[-\pi, \pi]$ with respect to the normalized Lebesgue measure (similarly for the corresponding norms) and N will be assumed to be large.

A well-known conjecture of Hardy and Littlewood asserts that if $c_1 = \cdots = c_N = 1$ then

$$\|F\|_1 > C \log N$$

(if the n_i's are in arithmetic progression then we have $\|F\|_1 \sim C \log N$). A method introduced by P. Cohen and further improved by H. Davenport and the author (see [1]) leads to the estimate

$$\|F\|_1 > C(\log N/\log\log N)^{1/2} \tag{1}$$

for any F with $|c_i| \geqslant 1$, $i = 1, \ldots, N$. (1) appears to be the best known result up to now.

Let $M = |\min_x f(x)|$. Since

$$2M = \int (2f + 2M) \geqslant \|2f\|_1 - 2M$$

and $2f(x)\exp(in_Nx)$ is of the same form as F with $2N$ terms, (1) implies

$$M > C(\log N/\log\log N)^{1/2}. \tag{2}$$

The case $c_1 = \cdots = c_N = 1$ of (2) has been proved by a method different than that of Cohen by K. F. Roth (see [2] where more information and references concerning these problems can be found). Again (2) appears to be the best known result concerning M. The example $2f(x) = |G|^2 - N$, where G has the

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