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DISCRETE SPECTRUM OF THE WEIL REPRESENTATION

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1. Weil representation. Let Q be a nondegenerate quadratic form on $\mathbf{R}^k$. Let $O(Q)$ be the orthogonal group of Q. One owes to A. Weil [4] the construction of a certain unitary representation π_Q of the group $\widetilde{\mathrm{Sl}_2} \times O(Q)$ in $L^2(\mathbf{R}^k)$, where $\widetilde{\mathrm{Sl}_2}$ is a two fold covering of $\mathrm{Sl}_2(\mathbf{R})$, i.e. given by pairs (g, ϵ) with $g \in \mathrm{Sl}_2(R)$ and $\epsilon = \pm 1$ satisfying the group law $(g, \epsilon)(g', \epsilon') = (gg', V(g, g')\,\epsilon\epsilon')$, where V is the Kubota cocycle on $\mathrm{Sl}_2(\mathbf{R})$ (with values in $\mathbf{Z}_2$). Let $w_0 \in \widetilde{\mathrm{Sl}_2}$ be the element $\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], -1\right)$. Then π_Q is given by

$$\text{(i)} \qquad \pi_Q(w_0)\varphi(X) = \delta_Q\hat{\varphi}(-M_Q(X)),\ \varphi \in L^2(\mathbf{R}^k),$$

where $M_Q \in \mathrm{Aut}(\mathbf{R}^k)$ so that $[X, M_Q(Y)] = Q(X, Y)$ for all $X, Y \in \mathbf{R}^k$ (with $[\ ,\ ]$ the usual dot product on $\mathbf{R}^k$) and $\delta_Q = |\det Q|^{-1/2}u_Q$ with u_Q a certain eighth root of unity determined explicitly in [2]. Moreover, ^ denotes the Fourier transform on $L^2(\mathbf{R}^k)$. Also we have

$$\text{(ii)} \qquad \pi_Q\left(\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, 1\right)\varphi(X) = |\alpha|^{k/2}\, e^{\sqrt{-1}\,\pi\beta\alpha Q(X,X)}\varphi(\alpha X), \quad \text{with } \alpha > 0$$

and

$$\text{(iii)} \qquad \pi_Q(g)\varphi(X) = \varphi(g^{-1}X) \quad \text{for } g \in O(Q).$$

Then (i), (ii), and (iii) determine π_Q explicitly. The main problem is to give a spectral decomposition of π_Q.

2. Discrete spectrum of π_Q. Let $\widetilde{K}$ be the maximal compact subgroup of $\widetilde{\mathrm{Sl}_2}$ given by

$$\left\{\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \epsilon\right) \Big| -\pi \leqslant \theta < \pi,\ \epsilon = \pm 1\right\}.$$

Then every unitary character of K is given by

$$k(\theta, \epsilon) \rightsquigarrow (\mathrm{sgn}\ \epsilon)^{2m}\, e^{\sqrt{-1}\, m\theta} \quad \text{with } m \in \tfrac{1}{2}\mathbf{Z}.$$

We let

$$A = \left\{a(r) = \left(\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}, 1\right) \Big| r > 0\right\}$$

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