

INADMISSIBLE RECURSION THEORY

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Let β be a limit ordinal, and let S_β be stage β of Jensen's S hierarchy for L (cf. [1, p. 82]). S_β is the setting for β -recursion theory, an extension of recursion theory on Σ_1 admissible, initial segments of L (so-called α -recursion theory) to rudimentarily closed, initial segments of L . A set $A \subset S_\beta$ is said to be β -recursively enumerable (β -r.e.) if it is Σ_1 definable over S_β . If β is Σ_1 admissible, i.e. S_β satisfies Σ_1 replacement, then many theorems about ordinary r.e. sets ($\beta = \omega$) remain true when r.e. is replaced by β -r.e. (cf. [3], [5] and [6]). The results below, particularly the solution to Post's problem devised by Friedman in Theorem 3, suggest that the assumption of Σ_1 admissibility is superfluous. This outcome is in the natural order of events, because arguments in the Σ_1 admissible case often consist of showing that the use of Σ_2 or Σ_3 admissibility when $\beta = \omega$ was unnecessary. It is a desirable outcome in that it implies that the methods of ordinary recursion theory can be applied to all levels of the J hierarchy for L (cf. [1]).

The definitions in this paragraph are drawn from α -recursion theory. A is β -recursive (β -rec.) if A and $S_\beta - A$ are β -r.e. A function f is β -rec. if its graph is. x is β -finite if $x \in S_\beta$. Let $F(e, x)$ be a Σ_1 formula such that $\{F(e, x) \mid e \in S_\beta\}$ is a list of all Σ_1 formulas with free variable x and parameters in S_β . For any $A \subset S_\beta$, $\{e\}^A(x)$ has y as a value if S_β satisfies

$$(Eu)(Ev)[u \subset A \ \& \ v \subset S_\beta - A \ \& \ F(e, \langle x, y, u, v \rangle)].$$

f is weakly β -recursive in A ($f \leq_{w\beta} A$) if for some e and all x , $\{e\}^A(x) = f(x)$. $B \leq_{w\beta} A$ if its characteristic function $C_B \leq_{w\beta} A$. Let B^* be the set of all β -finite $x \subset B$. B is β -recursive in A ($B \leq_\beta A$) if $B^* \leq_{w\beta} A$ and $(S_\beta - B)^* \leq_{w\beta} A$.

Suppose A is nonempty and β -r.e. In the admissible case an enumeration of A is a β -rec. map f of β onto A . This definition is unsuitable in the inadmissible case, because there may be a $\delta < \beta$ such that $f \upharpoonright \delta$, the range of f restricted to δ , is unbounded in S_β . A notion of enumeration which suits both cases is as follows. Let $G(x, y)$ be a Δ_0 formula such that $x \in A$ iff $S_\beta \models (Ey)G(x, y)$. Let A^δ be the set of all $x \in S_\delta$ such that $S_\delta \models (Ey)G(x, y)$. Then $\{A^\delta \mid \delta < \beta\}$ is an enumeration of A , namely a nondecreasing, β -recursive sequence (of length β) of β -finite sets whose union is A .

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