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BOOK REVIEWS

L. E. J. Brouwer collected works, Volume I, *Philosophy and foundations of mathematics*, A. Heyting, ed., North-Holland/American Elsevier, Amsterdam, New York, 1974, 628 + xv pp., $96.25.

This book contains the principal sources of intuitionistic logic and mathematics, Brouwer's particular brand of so-called constructive foundations (c.f.). To understand it, one must compare its merits and defects with those of other, better known versions of c.f. (including incidentally the bulk of Brouwer's early writings on c.f.).–To put first things first: Brouwer's final version is incomparably more imaginative. The commonplace versions are preoccupied with the business of 'pure' existence theorems $\exists x A(x)$ and the search for 'explicit' realizations t: $A(t)$. For the silent majority of mathematicians this business is hardly dramatic: there is nothing to stop one from presenting such t even if one does not reject pure existence theorems. What is more, mathematics has developed a whole arsenal of notions for stating significant differences between such t, much more pertinent than the crude idea of an 'explicit' t or the crude distinction between 'constructive' and 'nonconstructive' (definitions of) t.[1] Commonplace c.f. constitute a *restriction*, and thus form a proper part of ordinary mathematics–usually accompanied by grand, but dubious foundational (cl)aims, to which we return later on.

Brouwer's version of c.f. is *incomparable* with ordinary mathematics. On the one hand it does not contain higher set theory with the (transfinite) iteration of the power set operation applied to infinite sets. On the other it includes as principal objects of mathematical study (i) choice sequences of various kinds, for example, (the idealization of) the random sequences of throws of a die, and

[1] Specialists, for whom this and other footnotes are intended, may want some documentation. (i) A happy coincidence shows the *appreciation* by the Mathematical Establishment of significant 'explicit' realizations. Without much exaggeration: a Fields Medal was awarded in 1958, to Roth, for the 'pure' existence theorem $\forall n \exists q_0 \forall p \forall q(q > q_0 \to |\sqrt[3]{2} - p/q| > q^{-2-1/n})$, and another one in 1970, to Baker, for the 'worse' result $\exists q_0 \forall p \forall q(q > q_0 \to |\sqrt[3]{2} - p/q| > q^{-3+0.05})$ where, however, a (manageable) value for q_0 was supplied. So much for blind prejudice against an appropriate search for explicit realizations. (ii) In (topological) algebra, one asks whether for polynomials of odd degree, say a cubic with leading coefficient 1, a (real) *zero* is determined continuously in the coefficients. The answer is *No* for the field **R** with the usual topology (take $x^3 - 3x - c$); the answer is *Yes* for what Brouwer called *real number generators* (r.n.g.) such as binary expansions with the product topology provided the usual equivalence relation for r.n.g. need not be respected. (Quite generally, Brouwer's insistence on r.n.g. is appropriate when continuity is paramount: a continuous mapping from **R** into a discrete space is constant, but not for r.n.g.) (iii) In analysis, one asks about Brouwer's fixed point theorem, for the uniform convergence topology of mappings f of, say $S^2 \mapsto S^2$: Is there a continuous $\zeta: f \mapsto x \in S^2$ such that $f[\zeta(f)] = \zeta(f)$? The answer is *No* (also for r.n.g. x): approximations to a fixed point of f are not generally determined by approximations to f (and so one need not even ask if they are 'constructively' determined).—Once the attention of mathematicians is drawn to the ideas involved in (i)–(iii), their relevance is plain without any foundational preoccupation. Incidentally, several questions seem still open, for example: Are there topological versions of Hilbert's Nullstellensatz or of Artin's solution of Hilbert's 17th problem for complex (resp. real) coefficients (or their generators)?