BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

A STOCHASTIC MINIMUM PRINCIPLE

ROBERT J. ELLIOTT

Communicated by Alexandra Bellow, July 14, 1976

1. Pontrjagin's maximum principle [6] is a basic result in deterministic optimal control theory. Analogous results have been obtained for the optimal control of stochastic dynamical systems (see for example the survey by Fleming [3]), and a new approach to such problems, using the martingale theory of Meyer, was made in the paper of Davis and Varaiya [2]. In this paper, by observing that the cost function is a 'semimartingale speciale' (see [5]), we are able to simplify much of [2] and obtain quickly a very general dynamic programming minimum principle.

2. The dynamics are described by a stochastic differential equation

$$dx_t = f(t, x, u)dt + \sigma(t, x)dB_t \tag{2.1}$$

with initial condition $x(0) = x_0 \in R^m$. Here $t \in [0, 1]$, B is an m-dimensional Brownian motion and $x \in C$, the space of continuous functions from $[0, 1]$ to R^m. Write $F_t = \sigma\{x_s: s \leq t\}$ for the σ-field generated on C up to time t. The control values u are chosen from a compact metric space U. We suppose f and (nonsingular) σ satisfy the usual measurability and growth conditions (see [2]).

If an m-dimensional Brownian motion B_t on a probability space (Ω, μ) is given these conditions on σ ensure that the equation

$$x_t = x_0 + \int_0^t \sigma(s, x)dB_s$$

has a unique solution with sample paths in C.

DEFINITION 2.1. The admissible controls M_s^t over $[s, t] \subset [0, 1]$ are the measurable functions u: $[s, t] \times C \longrightarrow U$ (U is given the Borel σ-field) such that (i) for each τ, $s \leq \tau \leq t$, $u(\tau, \cdot)$ is F_τ measurable, (ii) for each $x \in C$, $u(\cdot, x)$ is Lebesgue measurable.

Such functions are nonanticipative feedback controls and the conditions on f ensure that for such a control $u(t, x) \in M_s^t$: $E[\exp[\xi_s^t(f^u)]\, F_s] = 1$ a.s. μ. Here $f^u(\tau, x) = f(\tau, x, u(\tau, x))$ and

$$\xi_s^t(f^u) = \int_s^t \{\sigma^{-1}(\tau, x)f^u(\tau, x)\}' dB_\tau - \frac{1}{2}\int_s^t |\sigma^{-1}(\tau, x)f^u(\tau, x)|^2 \, d\tau.$$

Writing $M = M_0^1$, for each $u \in M$ a measure μ_u is defined on (C, F_1) by

---

AMS (MOS) subject classifications (1970). Primary 93E20; Secondary 60H10.