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MONOTONICITY AND UPPER SEMICONTINUITY

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Introduction. We show in this note that set valued maximal monotone operators on a Hilbert space possess the upper semicontinuity property called property (Q), introduced by Cesari [2] and used extensively in the existence analysis of optimal control theory. As a particular consequence we conclude rather easily, the known result (see [1], for example) that maximal monotone operators have closed graph and are thus demiclosed. As a simple application of this to optimal control theory we give an existence theorem for a Mayer problem. Details and extensions are found in [5] where we study upper semicontinuity in the context of semiclosure operators of general topology.

Notations. Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$ and induced norm $\|\cdot\|$. Let 2^H denote the collection of all nonempty subsets of H. As in [1], a set valued function $F: H \longrightarrow 2^H$ is said to be maximal monotone, if its graph $G(F)$ is maximal with the property that $\langle y_2 - y_1, x_2 - x_1\rangle \geqslant 0$ for all $(x_1, y_1), (x_2, y_2) \in G(F)$. As in [2], $F: H \longrightarrow 2^H$ is said to have property (Q) if for each $x_0 \in H$,

$$F(x_0) = \bigcap_{\delta > 0} \operatorname{cl} \operatorname{co} \bigcup \{F(x), \|x - x_0\| < \delta\} \tag{1}$$

where cl co A denotes the (strong) closure of the convex hull of A. It is seen that if F is monotone then the right hand side of equation (1) is also monotone and hence we obtain

THEOREM 1. *If $F: H \longrightarrow 2^H$ is maximal monotone, then F has property (Q).*

REMARKS. 1. It is to be noted that maximality is important in the above theorem. For example, if $F(x) = \{[x]\}$, x real, where $[x]$ is the greatest integer $\leqslant x$, then F does not have property (Q) at $x = 0$. On the other hand, F is monotone but not maximal since $I + F$ is not surjective; indeed $3 \neq x + [x]$ for any real x.

2. $F(x)$ is closed and convex for each $x \in H$, if F has property (Q) and hence if F is maximal monotone.

3. Using Banach-Saks-Mazur theorem it is seen that if F has property (Q), $y_k \longrightarrow y_0$ weakly, $x_k \longrightarrow x_0$ strongly, and $y_k \in F(x_k)$, then $y_0 \in F(x_0)$. By

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