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EXISTENCE THEOREMS FOR STRONGLY COUPLED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS OVER BERNSTEIN CLASSES

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If f is an entire function of type σ, bounded on the real axis, a well-known theorem of Bernstein gives $\sup_{x\in\mathbf{R}} |f'(x)| \leqslant \sigma \sup_{x\in\mathbf{R}} |f(x)|$. Our objective is to construct a Banach space, based on this theorem, in which the existence problem for broad classes of partial differential equations can be solved with surprising ease. The idea of using Bernstein's theorem in the study of coupled systems is due to Nickel [3].

1. **Notation.** We denote real or complex m-dimensional space by $\mathbf{R}^m$ or $\mathbf{C}^m$ respectively, and the norm of a vector z in one of these spaces by $|z|$. The norm of any function $F\colon \mathbf{F} \longrightarrow \mathbf{C}^p$ is $\|F\| = \sup_{\xi\in\mathbf{F}} |F(\xi)|$. We choose $\sigma \in \mathbf{R}^m$ fixed, $\sigma_i \geqslant 0$, and use j as a multi-index; thus:

$$D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1}\,\partial x_2^{j_2}\cdots\partial x_m^{j_m}}, \qquad \sigma^j = \sigma_1^{j_1}\sigma_2^{j_2}\cdots\sigma_m^{j_m}.$$

2. **The class** $B(m, p, \sigma)$. The class $B(m, 1, \sigma)$ is the class of functions $f\colon \mathbf{R}^m \longrightarrow \mathbf{C}$ satisfying one of the following equivalent conditions:

(a) f can be extended to a holomorphic function $f^*\colon \mathbf{C}^m \longrightarrow \mathbf{C}$ so that, for all $z = x + iy \in \mathbf{C}^m$ and for some constant $M = M(f)$,

$$|f^*(z)| \leqslant M \exp(\sigma_1 |y_1| + \sigma_2 |y_2| + \cdots + \sigma_m |y_m|).$$

(b) f can be extended to a holomorphic function $f^*\colon \mathbf{C}^m \longrightarrow \mathbf{C}$ so that, for all $z = x + iy \in \mathbf{C}^m$ and all multi-indices j,

$$|D^j f^*(z)| \leqslant \|f\|\sigma^j \exp(\sigma_1 |y_1| + \sigma_2 |y_2| + \cdots + \sigma_m |y_m|) < \infty.$$

(c) $f \in C^\infty$ and $\|D^j f\| \leqslant \|f\|\sigma^j < \infty$ for all multi-indices j.

That (b) $\Rightarrow$ (c) $\wedge$ (a) is obvious, and (c) $\Rightarrow$ (b) by Taylor's formula. Also, (a) $\Rightarrow$ (b) follows for $m = 1$ from Bernstein's theorem and then in general by induction on m.

By (a), $B(m, 1, \sigma)$ is a linear space which contains all functions

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