BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

ON MONOTONE VS. NONMONOTONE INDUCTION

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Communicated by S. Feferman, May 11, 1976

1. **Introduction.** For definitions and notation in what follows, see [4] and [5]. If A is an infinite set and $\varphi(y_1 \cdots y_n, R, Y_1 \cdots Y_m) = \varphi(\bar{y}, R, \bar{Y})$ is a second order relation on A, we call φ *operative* if R is n-ary. For such a φ let

$$I_\varphi^\xi = \bigcup_{\eta<\xi} I_\varphi^\eta \cup \left\{(\bar{y}, \bar{Y}): \varphi\left(\bar{y}, \left\{\bar{y}: (\bar{y}, \bar{Y}) \in \bigcup_{\eta<\xi} I_\varphi^\eta\right\}, \bar{Y}\right)\right\} \quad \text{and} \quad I_\varphi = \bigcup_\xi I_\varphi^\xi.$$

If $\mathcal{F}$ is a collection of second order relations (for simplicity *collection of operators*) on A, then $\mathcal{F}$-IND^2 is the class of all second order relations of the form $\psi(\bar{x}, \bar{Y}) \Leftrightarrow I_\varphi(\bar{a}, \bar{x}, \bar{Y})$, for some operative $\varphi(\bar{u}, \bar{x}, R, \bar{Y})$ in $\mathcal{F}$ and constants $\bar{a}$ from A. As in [5] $\mathcal{F}$-IND is the class of all *relations* on A which are in $\mathcal{F}$-IND^2. We let $\mathcal{F}^{\text{mon}}$ be the collection of all operative $\varphi(\bar{y}, R, \bar{Y})$ in $\mathcal{F}$ which are *monotone* on R and we put $\neg\mathcal{F} = \{\neg\varphi: \varphi \in \mathcal{F}\}$. A collection of operators $\mathcal{F}$ on A is *adequate* if it contains all the $\Pi_1^0(\mathcal{C})$ second order relations, where $\mathcal{C}$ is a coding scheme on A and is closed under $\wedge$, $\vee$, $\exists^A$ and trivial combinatorial substitutions. Let $WF(S) \Leftrightarrow S$ be a well-founded relation on $A \Leftrightarrow \neg \exists a_0 a_1 a_2 \cdots \forall i (a_{i+1}, a_i) \in S$.

THEOREM 1. *Let $\mathcal{F}$ be an adequate collection of operators on an infinite set A. If $WF \in \neg\mathcal{F}$ and $\neg\mathcal{F} \subseteq \mathcal{F}^{\text{mon}}$-IND^2, then $\mathcal{F}$-IND^2 = $\mathcal{F}^{\text{mon}}$-IND^2.*

2. **Elementary induction.** Let $\mathcal{EL}$ be the collection of all the elementary second order relations on a structure $\mathfrak{A} = \langle A, R_1 \ldots R_l\rangle$ and let $\mathcal{EL}^+$ be the subcollection of $\mathcal{EL}^{\text{mon}}$ consisting of all operative $\varphi(\bar{x}, R, \bar{Y})$ which are definable by positive in R elementary formulas. One usually writes $\mathcal{EL}^+$-IND^2 = IND^2 and $\mathcal{EL}^+$-IND = IND. Clearly IND$^2 \subseteq \mathcal{EL}^{\text{mon}}$-IND$^2 \subset \mathcal{EL}$-IND^2 and it is well known that IND^2 is a tiny part of $\mathcal{EL}$-IND^2 for (say) almost acceptable $\mathfrak{A}$'s. By a basic result of Kleene and Spector for ω and Barwise-Gandy-Moschovakis in general (see [4, §8A]), on every *countable* almost acceptable structure, IND^2 = $\mathcal{EL}^{\text{mon}}$-IND^2 (= Π_1^1). On the other hand, letting $WF^n(S) \Leftrightarrow S$ is a 2n-ary relation on A which is well founded (viewed as binary on A^n), we have

COROLLARY 1. *Let $\mathfrak{A}$ be an infinite structure such that each WF^n is elementary. Then $\mathcal{EL}^{\text{mon}}$-IND^2 = $\mathcal{EL}$-IND^2.*

AMS (MOS) subject classifications (1970). Primary 02F27.