

ON L^1 CONVERGENCE OF CERTAIN COSINE SUMS

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Abstract. It is shown that to a certain cosine series f , a Rees-Stanojević cosine sum g_n can be associated such that g_n converges to f pointwise, and a necessary and sufficient condition for L^1 convergence of g_n to f is given. As a corollary to that result we have a generalization of the classical result of this kind. Other corollaries are given concerning the well-known integrability conditions.

This paper gives an analogue for modified cosine sums of the classical result concerning L^1 convergence of a Fourier sine series. Rees and Stanojević [1] introduced these cosine sums that approximate their pointwise limit "better" than the classical cosine series since they converge in the L^1 metric space to their limit when the classical cosine series may not.

LEMMA 1. Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ where $f_n(x) = \frac{1}{2}a(0) + \sum_{k=1}^n a(k)\cos kx$, $\lim_{n \rightarrow \infty} a(n) = 0$, and $\sum_{k=0}^{\infty} |\Delta a(k)| < \infty$. Let $g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a(k) + \sum_{k=1}^n \sum_{j=k}^n \Delta a(j)\cos kx$. Then $\lim_{n \rightarrow \infty} g_n(x) = f(x)$.

THEOREM 1. Let f , f_n , and g_n be as defined in Lemma 1. Then g_n converges to f in the L^1 metric if and only if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_0^\delta |\sum_{k=n+1}^{\infty} \Delta a(k)D_k(x)| < \epsilon$ for all $n \geq 0$, where $D_k(x)$ is the Dirichlet kernel.

COROLLARY 1. Let f_n and f be as defined in Lemma 1. If for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_0^\delta |\sum_{k=n}^{\infty} \Delta a(k)D_k(x)| < \epsilon$ for all $n \geq 0$ then f_n converges to f in the L^1 metric if and only if $\lim_{n \rightarrow \infty} a(n) \log n = 0$.

COROLLARY 2. Let f and g_n be as defined in Lemma 1. If $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n+1) < \infty$, then g_n converges to f in the L^1 metric.

COROLLARY 3. Let f and g_n be as defined in Lemma 1. If $\sum_{k=1}^{\infty} |\Delta a(k)| \log k < \infty$, then g_n converges to f in the L^1 metric.

COROLLARY 4. Let f and g_n be as defined in Lemma 1. If $a(n) = b(n) + c(n)$ where $\lim_{n \rightarrow \infty} b(n) = \lim_{n \rightarrow \infty} c(n) = 0$, $\sum_{n=1}^{\infty} |\Delta b(n)| \log n < \infty$, and $\sum_{n=1}^{\infty} |\Delta^2 c(n)|(n+1) < \infty$, then g_n converges to f in the L^1 metric.

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¹Portions of these results appear in a doctoral thesis of John W. Garrett at the University of Missouri-Rolla in 1974.