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DIRECT SUM PROPERTIES OF QUASI-INJECTIVE MODULES

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ABSTRACT. A functorial method is described by which certain problems can be transferred from quasi-injective modules to nonsingular injective modules. Applications include the uniqueness of nth roots: If A and B are quasi-injective modules such that $A^n \cong B^n$, then $A \cong B$.

All rings in this paper are associative with unit, all modules are unital right modules, and endomorphism rings act on the left. The letter R denotes a ring. We use $J(-)$ to denote the Jacobson radical.

Recall that a module A is *quasi-injective* provided any homomorphism of a submodule of A into A extends to an endomorphism of A. For example, all injective modules and all semisimple (completely reducible) modules are quasi-injective.

THEOREM 1. *Let A be a quasi-injective right R-module, and set $Q = \mathrm{End}_R(A)$. Then $Q/J(Q)$ is a regular, right self-injective ring, and idempotents can be lifted modulo $J(Q)$.*

PROOF. Regularity and idempotent-lifting were proved by Faith and Utumi [2, Theorems 3.1, 4.1]. Self-injectivity was proved by Osofsky [6, Theorem 12] and Renault [7, Corollaire 3.5]. □

PROPOSITION 2. *Let A be a quasi-injective right R-module, and set $Q = \mathrm{End}_R(A)$. Let $\mathfrak{A}$ denote the category of all direct summands of finite direct sums of copies of A, and let $\mathcal{P}$ denote the category of all finitely generated projective right $(Q/J(Q))$-modules. Then there exists an additive (covariant) functor $F: \mathfrak{A} \longrightarrow \mathcal{P}$ with the following properties.*

(a) *For all $B, C \in \mathfrak{A}$, the induced map $\mathrm{Hom}_{\mathfrak{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(F(B), F(C))$ is surjective.*

(b) *Given any $P \in \mathcal{P}$, there exists $B \in \mathfrak{A}$ such that $F(B) \cong P$.*

(c) *A map $f \in \mathfrak{A}$ is an isomorphism if and only if $F(f)$ is an isomorphism in $\mathcal{P}$.*

PROOF. If $\mathcal{P}_0$ denotes the category of all finitely generated projective right Q-modules, then $\mathrm{Hom}_R(A, -)$ defines a category equivalence $G: \mathfrak{A} \longrightarrow \mathcal{P}_0$. Second, $(-) \otimes_Q (Q/J(Q))$ gives us an additive functor $H: \mathcal{P}_0 \longrightarrow \mathcal{P}$, and we set $F = HG$.

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