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EIGENVALUES ASSOCIATED WITH A CLOSED GEODESIC[1]

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1. **Background.** Intuitive arguments drawn from quantum mechanics and optics suggest that there should be some relation between the closed geodesics (periodic "particles") on a compact Riemannian manifold X and the eigenvalues (periodic "waves") of the Laplace-Beltrami operator Δ_X. Indeed, in 1959, Huber [8] proved, for X a surface of constant negative curvature, that the set of lengths of closed geodesics on X and the spectrum of Δ_X determine one another. The relation given by Huber between these two sequences of numbers is sufficiently complicated to make it extremely difficult to find one sequence explicitly, given the other.

Recently, Colin de Verdière [2], then Chazarain [1] and Duistermaat and Guillemin [3] have shown that, for most Riemannian metrics on any differentiable manifold, the spectrum of the Laplacian determines the lengths of the closed geodesics and their Morse indices modulo 4. Here, the lengths of the closed geodesics appear as the singular points of the distribution

$$\hat{\sigma}(t) = \operatorname{Trace}(e^{it\sqrt{\Delta_X}}) = \sum_{\lambda_j \in \operatorname{Spec} \Delta_X} e^{i\sqrt{\lambda_j}\, t}$$

on the real line. This result, although very striking, leaves open some important questions. To apply it in any particular case, one would need to know a lot about Spec Δ_X to get any information about the closed geodesics; even then, a formidable calculation would be involved in all but the simplest cases. In fact, when one is "handed" a Riemannian manifold, it is more likely that one knows something about the closed geodesics than about the spectrum.

In quantum mechanics, the eigenvalues are energy levels, and any available information about them is of interest. In a series of four papers culminating in [6], Gutzwiller used the method of Feynman integrals to find a contribution to the spectral density distribution

$$\sigma(\lambda) = \sum_{\lambda_j \in \operatorname{Spec} \Delta_X} \delta(\lambda - \lambda_j)$$

corresponding to a single closed geodesic γ. For stable γ, this contribution is a series of δ-functions at locations which are presumably approximate eigenvalues;

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