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THE Q-MATRIX PROBLEM FOR MARKOV CHAINS

BY DAVID WILLIAMS

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1. Let I be a countable set. A ("standard") *Markov transition function* $(P(t))$ on I may be regarded as a family $\{p_{ij}(\cdot): i, j \in I\}$ of functions on $[0, \infty)$ such that (for $i, j \in I$ and $s, t \in [0, \infty)$)

$$p_{ij}(t) \geqslant 0, \quad \sum_{k \in I} p_{ik}(t) = 1,$$

$$p_{ij}(s+t) = \sum_{k \in I} p_{ik}(s) p_{kj}(t), \quad \lim_{u \downarrow 0} p_{ii}(u) = p_{ii}(0) = 1.$$

If $(P(t))$ is a Markov transition function on I, the (Doob-Kolmogorov) limits

$$-q_{ii} = \lim_{t \downarrow 0} t^{-1}[1 - p_{ii}(t)], \quad q_{ij} = \lim_{t \downarrow 0} t^{-1} p_{ij}(t)$$

exist in $[0, \infty]$ and satisfy

(DK1) $$0 \leqslant q_{ij} < \infty \quad (i \neq j),$$

(DK2) $$\sum_{k \neq i} q_{ik} \leqslant -q_{ii} \leqslant \infty.$$

The $I \times I$ matrix $Q = (q_{ij})$ is called the *Q-matrix* of $(P(t))$ and we write $Q = P'(0)$.

The following theorem solves the *Q-matrix problem* for the case when all states are instantaneous ($q_{ii} = -\infty$, $\forall i$).

THEOREM. *Let Q be an $I \times I$ matrix with*

(1) $$q_{ii} = -\infty \quad (\forall i); \qquad 0 \leqslant q_{ij} < \infty \quad (\forall i, j: i \neq j).$$

For Q to be the Q-matrix of a Markov transition function $(P(t))$, it is necessary and sufficient that the following conditions (2) *and* (3) *hold*:

(2) $$\sum_{j \notin \{a,b\}} q_{aj} \wedge q_{bj} < \infty \quad (\forall a, b: a \neq b);$$

(3) *for every finite subset H of I,*

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