BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

APPROXIMATE FUNCTIONAL COMPLEXITY

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Communicated by James H. Bramble, June 20, 1975

Hilbert's 13th problem dealt with the functional complexity of a specific function of three variables. One measure of the complexity of a function of n variables is whether it can be represented in terms of functions of fewer variables according to a specific schema. For example, we may ask if a given function $F(x, y)$ is nomographic—i.e. can it be written in the form $f(\phi(x) + \psi(y))$, using only functions of one variable. A larger class of functions are those that can be represented as uniform limits of nomographic functions. Membership in such a class is a measure of the *approximate* functional complexity of a function F, and may be an appropriate concept in discussing computational approximation.

Some of the important questions dealing with complexity have been answered by the work of Vitushkin [8], Arnol'd [1], and Kolmogorov [5]. In the present note, we describe some new techniques for discussing approximate complexity and state several theorems which extend the scope of earlier results.

We first observe that if all the component functions used in a particular representation schema are sufficiently smooth, then the resulting class of representable functions will be solutions of one or more specific partial differential equations, which may in fact yield local characterizations for smoothly representable functions. (For example, smoothly representable functions of the form $f(\phi(x, y), \psi(y, z))$ must satisfy a fourth-order equation with 55 terms.) However, this observation does not seem to be immediately useful in treating approximate representation, nor in dealing with representation by functions required only to be continuous. (It is tempting to hope that an appropriate concept of weak solution will be useful here.)

Our results are of two types. The first is based on the study of level sets, as with much of the preceding work in complexity; excellent surveys may be found in Sprecher [7], and in [2] and [6]. Any representation schema can be regarded as a mapping diagram whose commutativity imposes necessary conditions on each component function, which in turn give rise to relations between their level sets. In some cases, this may be carried over to *approximate* representation. The following is typical.

AMS (MOS) subject classifications (1970). Primary 41A30; Secondary 26A72.

[1]Research supported by NSF grant MPS75-06687 and ARO contract DA31-124AROD462.