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THE APPROACH OF SOLUTIONS OF NONLINEAR DIFFUSION EQUATIONS TO TRAVELLING WAVE SOLUTIONS

BY PAUL C. FIFE AND J. B. McLEOD[1]

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1. This note is concerned with the pure initial-value problem for the nonlinear diffusion equation

$$u_t - u_{xx} - f(u) = 0 \qquad (-\infty < x < \infty,\ t > 0), \tag{1}$$

with $u(x, 0) = \phi(x)$. This problem has attracted an increasing amount of attention in recent years, one of the central questions being whether or not the solution $u(x, t)$ tends as $t \to \infty$ to a travelling wave solution $U(x - ct)$. ([1] gives a general bibliography.) We adopt the usual normalization of the problem by assuming throughout that $f \in C^1[0, 1]$, $f(0) = f(1) = 0$, $0 \leqslant \phi \leqslant 1$, so that, as is well known, $0 \leqslant u(x, t) \leqslant 1$ for all x, t.

2. A typical convergence result that we can prove is the following.

THEOREM A. *Let $f \in C^1[0, 1]$, with $f(0) = f(1) = 0, f'(0) < 0, f(1) < 0$,*

$$f(u) < 0 \ \ \text{for } 0 < u < \alpha_0, \qquad f(u) > 0 \ \ \text{for } \alpha_1 < u < 1,$$

and assume that there exists a travelling wave solution $U(x - ct)$ with $U(-\infty) = 1$, $U(\infty) = 0$, $0 \leqslant U \leqslant 1$. Let ϕ satisfy $0 \leqslant \phi \leqslant 1$, $\liminf_{x\to-\infty} \phi(x) > \alpha_1$, $\limsup_{x\to\infty} \phi(x) < \alpha_0$. Then there exists some x_0 such that,

$$\lim_{t\to\infty} |u(x, t) - U(x - ct - x_0)| = 0$$

uniformly in x. If ϕ is monotonic, then the approach is in fact exponential.

We remark that such a travelling wave U can be shown to be necessarily monotonic, and it is an obvious consequence of Theorem A that U is unique up to translation. This can, of course, be shown directly (Theorem C below), and conditions under which U will exist are discussed in Theorem D.

In some cases the solution develops into a pair of diverging travelling waves, and this is relevent to the case where ϕ is of compact support.

THEOREM B. *Let f satisfy the hypotheses of Theorem* A, *and suppose $c >$*

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