BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

A PRIORI ESTIMATES, GEOMETRIC EFFECTS AND ASYMPTOTIC BEHAVIOR

BY FRITZ JOHN

Many physical phenomena can be described by a partial differential equation $Pu=0$. Here P denotes some differential operator or system of such operators, and u, the unknown function, is either a scalar or a vector. The differential equation connects the derivatives of u at each point of its domain D. The mathematician is interested in the *global* consequences of this *local* constraint, especially those leading to a better understanding of the physical processes described. There are many different angles from which to look at this subject. In this talk I intend to discuss some particular ones that have to do with the use of *a priori inequalities*, either *implied* by the differential equation or *postulated* for u. Some general remarks will lead up to their use.

Naturally, the first impulse of a mathematician on being confronted with an equation $Pu=0$ is to *solve* it. Usually,[1] the equation is not sufficient to determine u, and we have a whole infinite family S of solutions u. To characterize the individual members of S, we need additional pieces of information, *data* f taken from a family ϕ. Ideally, we generate the general element u of S by a continuous 1-1 mapping $T:\phi\rightarrow S$. The "problem" of constructing u from f is then "correctly-set" or "well-posed" for the equation $Pu=0$ in the sense of Hadamard [**3**], [**4**]. The best known example is the *Laplace equation* $\Delta u=0$ in a bounded domain D with continuous boundary values f prescribed on the boundary B of D. Such a mapping of data f onto solutions u not only has an esthetic appeal, but in many cases a physical interpretation as well, in which f and u somewhat play the roles of cause and effect, indicating perhaps some pre-established harmony between mathematics and the physical world.

Progress in the solution of well-posed problems has been spectacular. Still, this should not blind us to the fact that in applications the role of the well-posed problem is very limited (see [**5**]). In most cases, the assumption that we have adequate knowledge of the data f to generate u is fiction. Moreover, generating the solution u by solving some particular well-posed problem may contribute very little to a real *understanding* of the constraint $Pu=0$, beyond the insight that *nice* f produce *nice* u. One might even say that the better behaved the solution of a problem, the more likely it is to be devoid of qualitative features the mind can get hold of. To produce some

---

Josiah Willard Gibbs Lecture, given under the auspices of the American Mathematical Society, January 23, 1975. The research was supported in part by National Science Foundation grant NSF-GP-36992X2; received by the editors February 28, 1975.

AMS (*MOS*) *subject classifications* (1970). Primary 35B45, 35B40.

[1] But not always, as found by H. Lewy; see [**1**], [**2**].