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THE DECISION PROBLEM FOR RECURSIVELY ENUMERABLE DEGREES

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If I have any message today for mathematicians in general, it is that consideration of difficult problems can be useful even when the problem is at present beyond solution. The problem I will discuss is unlikely to be solved in the near future, but I hope to show how the study of it leads to many more accessible problems.

In order to state the problem, we need some definitions. To save words, we agree that *number* means *natural number* (nonnegative integer) and *set* means *set of numbers.*

A set A is *recursive* if there is an algorithm for determining whether any given number is in A. A set A is *recursive in* a set B if there is an algorithm by which we can decide whether any given number x is in A, provided we are supplied with answers to all questions we choose to ask of the form 'Is y in B?'.

As an example, let $A=\{2x : x \in B\}$. Then B is recursive in A; for $x \in B$ iff $2x \in A$. Also A is recursive in B; for $x \in A$ iff x is even and $\frac{1}{2}x \in B$. (All this is independent of the choice of B.)

Writing $A \leqq_R B$ for A is recursive in B, we easily see that

$$A \leqq_R A, \tag{1}$$

$$A \leqq_R B \;\&\; B \leqq_R C \rightarrow A \leqq_R C. \tag{2}$$

Of course $A \leqq_R B \;\&\; B \leqq_R A$ does not imply $A=B$, as the above example shows. However, if we define

$$A \sim B \text{ iff } A \leqq_R B \;\&\; B \leqq_R A, \tag{3}$$

then (1) and (2) show that $\sim$ is an equivalence relation. The equivalence class of A is the *degree* of A; it is written $\mathrm{dg}(A)$. Setting

$$\mathrm{dg}(A) \leqq \mathrm{dg}(B) \text{ iff } A \leqq_R B, \tag{4}$$

we see from (1) and (2) that the set of degrees is a partially ordered set D.

The study of degrees was initiated by Kleene and Post [**2**] who observed two simple facts. (A) There is a smallest degree 0; it is the degree of every recursive set. (B) Every pair a, b of degrees has a least upper bound $a \cup b$. In fact, $\mathrm{dg}(A) \cup \mathrm{dg}(B) = \mathrm{dg}(A \oplus B)$, where $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$.

The main content of [**2**] and several subsequent papers is that D has very few nice properties other than (A) and (B). For example, it is shown in [**2**] that D is not a linearly ordered set, or even a lattice. In [**11**] it is shown that no strictly increasing sequence of degrees has a least upper bound.

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