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THE ANALYTIC THEORY OF ALGEBRAIC NUMBERS

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1. **The basics of algebraic number theory.** An algebraic number field is a field $K=\boldsymbol{Q}(\alpha)$ where α is a zero of an irreducible (over $\boldsymbol{Q}$) polynomial $f(x)$ with integral coefficients. The degree of K, which we denote by $n=n(K)=[K:\boldsymbol{Q}]$, is the degree of $f(x)$. We write the roots of $f(x)=0$ as $\alpha^{(1)}$, $\alpha^{(2)}, \cdots, \alpha^{(n)}$ in such a way that for $1 \leqq j \leqq r_1=r_1(K)$, $\alpha^{(j)}$ is real, while for $j>r_1$, $\alpha^{(j)}$ is complex. If we let $n=r_1+2r_2$, then it is customary to order the $r_2=r_2(K)$ complex conjugate pairs of roots so that for $r_1+1 \leqq j \leqq r_1+r_2$, $\overline{\alpha^{(j)}}=\alpha^{(j+r_2)}$. The $\alpha^{(j)}$'s are called the conjugates of α and the fields $K^{(j)}=\boldsymbol{Q}(\alpha^{(j)})$ are called the conjugate fields of K. If $r_2=0$, we say K is totally real and if $r_1=0$, we say K is totally complex.

The integers of K are those elements of K which are zeros of a polynomial with integer coefficients and leading coefficient 1. The integers of K form a ring which we denote by $\mathfrak{o}$. As is well known, factorization of the integers of K into prime integers is not necessarily unique. Various equivalent ways of remedying this have been used; we follow Dedekind's method. If $\alpha_1, \cdots, \alpha_k$ are elements of K, the set

$$\mathfrak{a}=[\alpha_1, \cdots, \alpha_k]=\left\{\sum_{i=1}^{k} a_i\alpha_i \mid a_i \in \boldsymbol{Z}\right\}$$

is called the module generated by $\alpha_1, \cdots, \alpha_k$ (today it would be called a finitely generated $\boldsymbol{Z}$ module). The ring $\mathfrak{o}$ is an example of such a module; on the other hand, K is not an example since it is not finitely generated over $\boldsymbol{Z}$. If $\mathfrak{b}=[\beta_1, \cdots, \beta_m]$ is another module, we define the product $\mathfrak{ab}$ to be the module generated by the km numbers $\alpha_i\beta_j$.

Since 1 is in $\mathfrak{o}$, we always have $\mathfrak{oa} \supset \mathfrak{a}$ for any module $\mathfrak{a}$. If $\mathfrak{oa}=\mathfrak{a}$ then we say $\mathfrak{a}$ is a fractional ideal of K. The nonzero fractional ideals of K form an abelian group under multiplication with identity element $\mathfrak{o}$. An integral ideal, or just ideal for short, is a fractional ideal of K which is contained in $\mathfrak{o}$. The integral ideals of K are precisely the ideals of $\mathfrak{o}$ in the sense of ring theory today. Every fractional ideal is a quotient of two integral ideals and factorization of ideals into prime ideals is unique.

Among the fractional ideals of K are the principal fractional ideals. If α is in K then the principal fractional ideal generated by α is

$$(\alpha)=\alpha\mathfrak{o}=[\alpha]\mathfrak{o}.$$

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