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PROOF OF A CONJECTURE OF ASKEY ON ORTHOGONAL EXPANSIONS WITH POSITIVE COEFFICIENTS

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We prove the following theorem, which was conjectured by R. Askey [1], [2].

THEOREM 1. *Let* $\{p_n(x)\}$ *and* $\{q_n(x)\}$ *be polynomials orthogonal over* $[0, \infty)$ *with respect to distributions* $du(x)$ *and* $x^c\,du(x)$ $(c > 0)$, *respectively, normalized so that*

$$p_i(0) > 0, \quad q_i(0) > 0, \qquad i = 0, 1, \ldots. \tag{1}$$

Then the coefficients $\{a_{jn}\}$ *in the expansions*

$$q_n(x) = \sum_{j=0}^{n} a_{jn} p_j(x), \qquad n = 0, 1, \ldots \tag{2}$$

are all positive.

It is to be understood here that $du(x)$ has moments of all orders on $[0, \infty)$, and that $n \leqslant N - 1$ if $du(x)$ is a discrete distribution over only N points.

Using different arguments, Askey [1] and the author [4] have shown that the conclusion of this theorem follows from classical results if c is a positive integer, and, allowing for a difference in normalization, results obtained in [4] imply also that the coefficients in the "inverse" expansion, $p_n(x) = \Sigma_{j=0}^n b_{jn} q_j(x)$, satisfy $(-1)^{n-j} b_{jn} \geqslant 0$ if c is a positive integer, but that this does not hold for all positive c.

It suffices to prove Theorem 1 for $0 < c < 1$, since its conclusion follows for any positive c from finitely many successive applications of this restricted result. We confine our attention to this case.

The roots of $p_i(x)$ and $q_i(x)$ are positive; therefore, (1) and Descartes' rule of signs imply that

$$(-1)^r p_i^{(r)}(0) \geqslant 0, \quad (-1)^r q_i^{(r)}(0) \geqslant 0, \qquad 0 \leqslant r \leqslant i. \tag{3}$$

This is needed below.

LEMMA. *Under the assumptions of Theorem* 1,

$$(-1)^j a(a-1) \cdots (a-j+1) \int_0^\infty x^a p_j(x)\, du(x) > 0 \tag{4}$$

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