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SMALL EIGENVALUES OF THE LAPLACE OPERATOR ON COMPACT RIEMANN SURFACES[1]

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Let $\mathscr{S}$ be a compact Riemann surface, which we will assume to have curvature normalized to be -1, and let $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots$ be the eigenvalues corresponding to the problem $\Delta F+\lambda F=0$ on $\mathscr{S}$, where Δ is the Laplacian for $\mathscr{S}$. In an otherwise very interesting and useful paper [**2**], McKean has stated that it is always the case that $\lambda_1\geqq\frac{1}{4}$. In this paper, we will show that this need not be true, and that in fact, it is possible to have as many λ_n's in $(0, \frac{1}{4})$ as one wishes. The existence of such λ_n's is of considerable interest, since they figure explicitly in the finer points of the theory of the distribution of the lengths of shortest closed geodesics in free homotopy classes on $\mathscr{S}$ (cf. [**1**]), as well as in the Riemann hypothesis for the Selberg zeta function corresponding to the trivial character on the fundamental group Γ of $\mathscr{S}$.

Our point of departure will be the version of the Selberg trace formula [**3**] appropriate to the case at hand, which runs as follows:

Suppose $\mathscr{S}$ is the quotient of the upper half-plane by the discontinuous group Γ, consisting of, apart from the identity, hyperbolic transformations. Let χ be a character of Γ, and let $0\leqq\lambda_0(\chi)\leqq\lambda_1(\chi)\leqq\cdots$ be the sequence of eigenvalues corresponding to the problem $\Delta F+\lambda F=0$ on $\mathscr{S}$, where the eigenfunction $F(x)$ is required to transform under Γ by $F(\gamma x)=\chi(\gamma)F(x)$. The discussion in [**3**] assures that the $\lambda_n(\chi)$'s are real and $\geqq 0$, and that the set of such eigenfunctions is complete in the space consisting of those measurable functions on the upper half-plane which transform in this manner, and which are L^2 over a fundamental domain of Γ. Now suppose $h(z)$ is an even function, holomorphic in a strip of the form $|\operatorname{Im} z|<\frac{1}{2}+\varepsilon$ $(\varepsilon>0)$, and satisfying a growth condition of the form $|h(z)|=O(1+|z|^2)^{-1-\varepsilon}$, uniformly in the strip. Associate with the sequence $\lambda_0(\chi), \lambda_1(\chi), \cdots$ of eigenvalues, a sequence R, consisting of those numbers $r(\chi)$ that satisfy the equations $\lambda_n(\chi)=\frac{1}{4}+r^2(\chi)$ $(n=0, 1, 2, \cdots)$. Apart

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[1] Professor Selberg has pointed out to me that he has previously obtained this and similar results by, among other things, a technique that directly establishes the continuous dependence of the spectrum on the character.