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ZEROS OF DERIVATIVE OF RIEMANN'S ξ-FUNCTION

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Riemann's ξ-function is defined by $\xi(s)=H(s)\zeta(s)$ where $\zeta(s)$ is the Riemann zeta-function and $H(s)=\frac{1}{2}(s^2-s)\pi^{-s/2}\Gamma(s/2)$. The functional equation is $\xi(s)=\xi(1-s)$. Moreover ξ is an entire function which has as its zeros precisely those of ζ in the critical strip. Because $\xi(\frac{1}{2}+it)$ is real, it follows that between consecutive zeros of ξ on the $\frac{1}{2}$-line there is at least one zero of ξ'.

It has recently been shown [1], [2] that $\zeta(s)$ has at least $\frac{1}{3}$ of its zeros in the critical strip on $\sigma=\frac{1}{2}$. Here a similar result will be proved for $\xi'(s)$ (for which $\frac{1}{3}$ is already implied by the remark above). Let $U=T/\log^{10} T$. Then the following theorem will be sketched.

THEOREM. *More than $\frac{7}{10}$ of the zeros of $\xi'(s)$ in $T<t<T+U$ occur on $\sigma=\frac{1}{2}$.*

By Stirling's formula, for $|\sigma|<10$, $H(s)=e^{F(s)}$, where

$$F'(s) = \tfrac{1}{2}\log s/2\pi + O(1/s), \qquad F''(s) = O(1/s).$$

From $\xi(s)=H(s)\zeta(s)=H(1-s)\zeta(1-s)$ follows

$$\begin{aligned} \xi'(s) &= H'(s)\zeta(s) + H(s)\zeta'(s) \\ &= -H'(1-s)\zeta(1-s) - H(1-s)\zeta'(1-s), \end{aligned} \tag{1}$$

and also

$$\begin{aligned} &H''(s)\zeta(s) + 2H'(s)\zeta'(s) + H(s)\zeta''(s) \\ &\qquad = H''(1-s)\zeta(1-s) + 2H'(1-s)\zeta'(1-s) + H(1-s)\zeta''(1-s). \end{aligned}$$

Since $H'=HF'$, and $H''=H'F'+HF''$,

$$\begin{aligned} &F'(s)[H'(s)\zeta(s) + H(s)\zeta'(s)] \\ &\qquad\qquad - F'(1-s)[H'(1-s)\zeta(1-s) + H(1-s)\zeta'(1-s)] \\ &\quad = -H'(s)\zeta'(s) - H(s)\zeta''(s) - H(s)F''(s)\zeta(s) \\ &\qquad + H'(1-s)\zeta'(1-s) + H(1-s)\zeta''(1-s) \\ &\qquad + H(1-s)F''(1-s)\zeta(1-s). \end{aligned}$$

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