BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

CONDITIONS FOR A UNIVERSAL MAPPING OF ALGEBRAS TO BE A MONOMORPHISM

BY JOHN L. MACDONALD

Communicated by Saunders Mac Lane, January 3, 1974

Introduction. We give necessary and sufficient conditions for the monicity of any given unit morphism associated with an algebraic functor and its left adjoint. The direct verification of these conditions in specific cases can be somewhat subtle. However, a stronger set of sufficient conditions for monicity may be given which is easier to check directly. The latter conditions are still sufficiently general to provide a categorical form for the proof of the Birkhoff Witt theorem, closely related to Birkhoff's original proof [3], as well as one for the Schreier theorem on free products of groups with amalgamated subgroups [1], [5].

1. **Necessary and sufficient conditions.** We consider algebras defined by a set Ω of operators and a set E of identities as in Mac Lane [4]. The diagram $VU:\mathscr{A}\to\mathscr{B}\to\mathscr{D}$ is called a *standard diagram of algebras* if

1. $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{D}$ are the categories of $\langle\Omega, E\rangle$, $\langle\Omega', E'\rangle$ and $\langle\Omega'', E''\rangle$ algebras, respectively, with $\Omega''\subseteq\Omega'$ and $E''\subseteq E'$, and
2. V is the forgetful functor on operators $\Omega'-\Omega''$ and identities $E'-E''$ and U is a functor commuting with the underlying set functors on $\mathscr{A}$ and $\mathscr{B}$. Note that U is *not* necessarily a functor forgetting part of Ω and E.

We next describe a functor $C_V:\mathscr{B}\to\mathrm{Grph}$ associated to each pair consisting of a standard diagram VU of algebras and an adjunction $\langle L, VU, \phi'\rangle:\mathscr{D}\to\mathscr{A}$, where Grph is the category of directed graphs in the sense of [4]. Given $G\in|\mathscr{B}|$ the objects of the graph $C_V(G)$ are the elements of the underlying set $|LVG|$ of LVG and its arrows are described recursively by:

1. $\omega_{ULVG}(|\eta'_{VG}|x_1, \cdots, |\eta'_{VG}|x_n)\to|\eta'_{VG}|\omega_G(x_1, \cdots, x_n)$ is an arrow if ω is in the set $\Omega'-\Omega''$ of operators forgotten by V and $|\eta'_{VG}|:|G|\to|LVG|$ is the set map underlying the unit $\eta'_{VG}:VG\to VULVG$ of the adjunction $\langle L, VU, \phi'\rangle$ and $(x_1, \cdots, x_n)$ is an n-tuple of elements of $|G|$ for which $\omega_G(x_1, \cdots, x_n)$ is defined.

AMS (MOS) subject classifications (1970). Primary 18B15, 18C10, 08A25; Secondary 17B35, 20E30.