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ON THE EXTENSION OF BASIC SEQUENCES TO BASES

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ABSTRACT We show that there exists a subspace G with a basis of some Banach space E with a basis, such that no basis of G can be extended to a basis of E.

A sequence $\{x_n\}$ in a (real or complex, infinite dimensional) Banach space E is called (a) a *basis* of E, if for every $x \in E$ there is a unique sequence of scalars $\{\alpha_n\}$ such that $x=\sum_{i=1}^{\infty} \alpha_i x_i$; (b) a *basic sequence* if $\{x_n\}$ is a basis of its closed linear span $[x_n]$ in E. The following problem was raised by A. Pełczynski (see [5] or [7, p. 27, Problem 4.1]): Let $\{y_n\}$ be a basic sequence in a Banach space E with a basis. Does there exist a basis $\{x_n\}$ of E with the property that for each n there is an index i_n such that $x_{i_n}=y_n$? Or, in other words, can $\{y_n\}$ be extended to a basis of E?

A. Pełczyński and H. P. Rosenthal have communicated to us that recently they have solved this problem in the negative, for $E=L^p([0, 1])$ $(2<p<\infty)$ and $E=L^1([0, 1])$ [6]. However, since in their counterexamples $\{y_n\}$ had some permutation $\{y_{\sigma(n)}\}$ which can be extended to a basis of E, they have raised the problem whether there exists a basic sequence $\{y_n\}$ in some Banach space E with a basis, such that no permutation $\{y_{\sigma(n)}\}$ of $\{y_n\}$ can be extended to a basis of E. In the present note we shall show even more, namely, that *there exists a subspace G with a basis of some Banach space E with a basis, such that no basis of G can be extended to a basis of E.* Our proof is very short, but uses deep results of Enflo [1], Lindenstrauss [4] and Johnson-Rosenthal-Zippin [3].

EXAMPLE. Let F be a separable Banach space which has no basis [1]. By [4] there exists a separable Banach space B such that the conjugate space B^* has a shrinking basis and that $B^{**}/\pi(B)$ is isomorphic to F, where π is the canonical embedding of B into B^{**}. Then B^{**} has a basis (see e.g. [7, Theorem 4.2, p. 272]) and by [3, Theorem 1.4(a)], B has a shrinking basis, so $\pi(B)$ has a shrinking basis. However, no basis $\{y_n\}$ of $G=\pi(B)$ can be extended to a basis $\{y_n\}\cup\{y'_n\}$ of $E=B^{**}$, since otherwise the quotient space $E/G=B^{**}/\pi(B)$ would have a basis, namely $\{\omega(y'_n)\}$,

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