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HILBERT SPACE AND VARIATIONAL METHODS FOR SINGULAR SELFADJOINT SYSTEMS OF DIFFERENTIAL EQUATIONS[1]

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1. **Introduction.** Let p, q, and r be certain $n\times n$ matrix-valued functions with real elements defined almost everywhere on the interval (a, b). Assume r is positive definite a.e. in $[a, b]$. We are going to consider a second order selfadjoint system of ordinary differential equations described by

$$(r(t)\dot{x}(t) + q^*(t)x(t))' = (q(t)\dot{x}(t) + p(t)x(t)), \tag{1.1}$$

where x lies in a certain class of n-vector valued functions on $[a, b]$. The differential equation (1.1) is said to be *singular* at a point t in $[a, b]$ if $r(t)$ is not positive definite. In this paper we restrict our attention to one singularity at $t=a$. However, much of the results carry over if we consider infinitely many singularities to the extent that the Lebesgue measure of the set of singular points is zero.

(1.1) is the Euler equation for the quadratic functional

$$\begin{aligned} J(x) &= x^*(c)kx(c) \\ &\quad + \int_a^b \{\dot{x}^*(t)r(t)\dot{x}(t) + 2x^*(t)q(t)\dot{x}(t) + x^*(t)p(t)x(t)\}\, dt, \end{aligned} \tag{1.2}$$

where k is an $n\times n$ symmetric matrix with c in (a, b) and $r(c)$ positive definite. Under appropriate conditions, J is well defined on the Hilbert space A of functions x defined on $(a, b]$ and absolutely continuous on each closed subinterval of $(a, b]$ with $\dot{x}^*R\dot{x}$ Lebesgue integrable on $(a, b]$. The inner product on A is given by

$$(x, y) = x^*(c)y(c) + \int_a^b \dot{x}^*(t)R(t)\dot{x}(t)\, dt, \tag{1.3}$$

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