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A REMARK CONCERNING PERFECT SPLINES

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Let $\boldsymbol{x}:=(x_i)$ be nondecreasing. For a sufficiently smooth f, denote by $f|_{\boldsymbol{x}}:=(f_i)$ the corresponding sequence given by the rule

$$f_i := f^{(j)}(x_i) \quad \text{with } j = j(i) := \max\{m \mid x_{i-m} = x_i\}.$$

Assuming that $\boldsymbol{x}$ is in $[a, b]$ and that $x_i < x_{i+n}$, all i, $f|_{\boldsymbol{x}}$ is defined for every f in the Sobolev space

$$W_\infty^{(n)}[a, b] := \{f \in C^{(n-1)}[a, b] \mid f^{(n-1)} \text{ abs. cont.}; f^{(n)} \in L_\infty[a, b]\}.$$

Karlin [6] discusses the problem of minimizing $\|f^{(n)}\|_\infty$ over all f in $\Pi(\boldsymbol{x}, \boldsymbol{\alpha}):=\{f \in W_\infty^{(n)} \mid f|_{\boldsymbol{x}}=\boldsymbol{\alpha}\}$ for a given sequence $\boldsymbol{\alpha}$, and announces the following

THEOREM (S. KARLIN [6]). *Let $\boldsymbol{x}=(x_i)_1^{n+r}$ be a given nondecreasing sequence in the finite interval $[a, b]$, with $x_i < x_{i+n}$, all i. Let $\boldsymbol{\alpha} \in \boldsymbol{R}^{n+r}$ be given. Then $\Pi(\boldsymbol{x}, \boldsymbol{\alpha})$ contains a perfect spline of order n with less than r (interior) knots, i.e., a function of the form*

$$p(x) = \sum_{i=0}^{n-1} a_i x^i + c\left[x^n + 2\sum_{i=1}^{k-1}(-)^i(x - \xi_i)_+^n\right] \tag{1}$$

for some real constants $a_0, \cdots, a_{n-1}$, and c, and for $a < \xi_1 < \cdots < \xi_{k-1} < b$ with $k \leqq r$. Further, $\|f^{(n)}\|_\infty$ takes on its minimum value over $f \in \Pi(\boldsymbol{x}, \boldsymbol{\alpha})$ at this p.

It is the purpose of this note to outline a simple proof of this theorem.

For this, denote by $[x_i, \cdots, x_{i+n}]f$ the nth divided difference of f at the $n+1$ points $x_i, \cdots, x_{i+n}$. Then $[x_i, \cdots, x_{i+n}](f-g)=0$ for all f, $g \in \Pi(\boldsymbol{x}, \boldsymbol{\alpha})$ and $i=1, \cdots, r$. Further, it is well known (see e.g., [2]) that, for $f \in W_1^{(n)}[a, b]$,

$$[x_i, \cdots, x_{i+n}]f = \int_a^b \varphi_i(t) f^{(n)}(t)\, dt$$

with

$$\varphi_i(t) := M_{i,n}(t)/n! := [x_i, \cdots, x_{i+n}](\cdot - t)_+^{n-1}/(n-1)!$$

a (polynomial) B-spline of order n having the knots $x_i, \cdots, x_{i+n}$. Hence,

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