BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

A NECESSARY AND SUFFICIENT CONDITION FOR LOWER SEMICONTINUITY[1]

BY LAMBERTO CESARI

Communicated by M. Protter, August 1, 1973

In the present paper we state a few lower semicontinuity and lower closure theorems we have recently proved in connection with weak convergence. In particular, for simple integrals of the calculus of variations and weak convergence, we prove here that the convexity of the integrand with respect to the derivatives is a necessary and sufficient condition for lower semicontinuity.

1. **Simple integrals of the calculus of variations.** In this section we consider only functionals of the form

$$I[x] = \int_a^b f_0(t, x(t), x'(t))\, dt, \qquad (') = d/dt. \tag{1}$$

We denote by A a given closed subset of the tx-space E^{n+1} ($t \in E^1$, $x=(x^1, \cdots, x^n) \in E^n$), and for the sake of simplicity we assume in this section that $A=\mathrm{cl}(\mathrm{int}\, A)$. Here, $f_0(t, x, u)$ is a given function, f_0: $A \times E^n \to E^1$, which again in this section will be assumed to be continuous on $A \times E^n$. Then the functional $I: \tau \to [-\infty, +\infty]$ is defined in the class τ of all absolutely continuous functions $x(t)=(x^1, \cdots, x^n)$, $a \leqq t \leqq b$, whose graph lies in A, and for which the measurable function $f_0(t, x(t), x'(t))$, $a \leqq t \leqq b$, has a Lebesgue integral on $[a, b]$ (finite, or $+\infty$, or $-\infty$).

We just note here that $\mathscr{F}: x \to f_0(\cdot, x(\cdot), x'(\cdot))$ is a Carathéodory, or Nemitskii operator [7].

For a sequence $x_k(t)=(x_k^1, \cdots, x_k^n)$, $a \leqq t \leqq b$, $k=1, 2, \cdots$, of absolute continuous functions we consider here the mode of convergence defined by (m) x_k' converges weakly in $L_1[a, b]$ and x_k converges uniformly in $[a, b]$. This mode of convergence is justified by the simple remark that, if x_k' converges weakly in $L_1[a, b]$ and, for at least one $\bar{t} \in [a, b]$, $x_k(\bar{t})$ converges (in E^n), then x_k converges uniformly in $[a, b]$. This remark is a simple consequence of the Dunford-Pettis theorem [**6**].

The interest of this mode of convergence lies in the fact that a *necessary and sufficient* condition for the lower semicontinuity of $I[x]$ at every

AMS (MOS) subject classifications (1970). Primary 49A25, 49A50; Secondary 49A05.

[1] This research was partially supported by AFOSR Research Project 71-2122 at the University of Michigan.