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POSITIVE NEAR-APPROXIMANTS AND SOME PROBLEMS OF HALMOS

BY RICHARD BOULDIN

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In [5] P. R. Halmos called for an investigation of those nonnegative operators P with the property that the distance from P to a fixed operator T is the same as the distance of T to the set of nonnegative operators. Such a P is a "positive approximant" of T. Halmos asked for the properties of such best approximating nonnegative operators when other norms besides the operator norm were used to compute distance. If $T=B+iC$, with $B=B^*$, $C=C^*$, then the formula $|||T|||^2=\|B^2+C^2\|$ defines a norm on the bounded operators with the pıoperty that

$$\|T\| \geqq |||T||| \geqq w(T) \geqq \tfrac{1}{2}\|T\|$$

where $w(T)$ denotes the numerical radius of T. The distance from T to the nonnegative operators is the same whether it is computed with the operator norm or with the new norm. A nonnegative operator which best approximates T in the new norm is a "positive near-approximant." This name is motivated by the facts that every positive approximant is a positive near-approximant and a positive near-approximant frequently turns out to be a positive approximant, although that is not necessarily the case.

P. R. Halmos gave an ingenious argument which resulted in a device for computing the distance of T to the nonnegative operators, denoted $\delta(T)$, and in a formula which defines a positive approximant of T for any T. If $T=B+iC$, with $B=B^*$, $C=C^*$, then the Halmos positive approximant is $P_0=B+(\delta^2-C^2)^{1/2}$ where $\delta=\delta(T)$. In [1] we showed that P_0 is absolutely maximal for the positive approximants of T, that is $P\leqq P_0$ whenever $P\in\mathscr{P}(T)$ with $\mathscr{P}(T)$ denoting the positive approximants; we used this fact as a basis for constructing positive approximants. In [2] we showed that P_0 is absolutely maximal for the positive near-approximants of T, denoted $\mathscr{P}'(T)$, and from this we constructed positive near-approximants. We have now carried this approach to the point of

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