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FINITE SUBGROUPS OF FINITE DIMENSIONAL DIVISION ALGEBRAS

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Let D be a finite dimensional division algebra with center K and let G be a finite odd order subgroup of the multiplicative group D^* of D. This note is concerned with the following:

CONJECTURE. If K contains no nonidentity odd order roots of unity, then G is cyclic.

We announce here some results and raise several questions about this conjecture. In [3] and [4] we proved this conjecture if K is either an algebraic number field or the completion of an algebraic number field. (The converse is also proved in [4]; if K is an algebraic number field which does contain an odd order nonidentity root of unity, then there is a finite dimensional division algebra central over K containing a noncyclic odd order subgroup.) In this note we will consider the more general case where K is an arbitrary field of characteristic zero.

By a K-division ring we mean a finite dimensional division algebra with center K. Let G be a finite subgroup of the multiplicative group of a K-division ring D and, for L a subfield of D, denote by $\mathscr{V}_L(G)$ the division subring of D generated by L and G. Let $\mathscr{Z}_L$ denote the center of $\mathscr{V}_L(G)$ and e_L the exponent of $\mathscr{V}_L(G)$. The following result is basic to our approach to the above conjecture:

THEOREM 1. *With notation as above, let ζ be a primitive e_Lth root of unity and let ϕ be an L-automorphism of $\mathscr{V}_L(G)$. Then $\phi(\zeta)=\zeta$.*

The proof of Theorem 1 involves an explicit computation using the description of $\mathscr{V}_Q(G)$ given by Amitsur in [2], where Q denotes the rational field.

Suppose G is a finite subgroup of the K-division ring D. Then $C_D(\mathscr{Z}_K)\simeq \mathscr{V}_K(G)\otimes_{\mathscr{Z}_K} A$ where A is a $\mathscr{Z}_K$-division ring and $C_D(\mathscr{Z}_K)$ denotes the centralizer in D of $\mathscr{Z}_K$. Suppose $\zeta \notin K$ where ζ is a primitive e_Kth root of unity. Then there is an automorphism ϕ of $C_D(\mathscr{Z}_K)$ with $\phi(\zeta)\neq\zeta$.

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