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# EXPONENTIAL ACTION OF A PENDULUM

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ABSTRACT. The total change in the action of a slowly modulated oscillator is transcendentally small in the slowness parameter, if the modulation is smooth. To approximate it thus requires methods for bypassing the whole asymptotic expansion of the oscillator. A theorem on the exponential property of the action for analytic frequency variation and an outline of an elementary proof are reported. It rests on an asymptotic split of the angle variable into its algebraic and exponential parts.

1. **Introduction.** The amplitude of a pendulum of slowly changing length has an asymptotic expansion in the small parameter $\varepsilon$, but the action is an 'adiabatic invariant': its total change has the zero expansion, if the length changes smoothly enough. A host of asymptotic methods have been developed for such slowly modulated oscillators—stationary phase, WKB, two-timing, etc.—but all lead to asymptotic expansions and are helpless when an approximation to a transcendental property is needed.

To promote development of methods for bypassing expansions to strike directly at subdominant properties, we report an extension of the theorem of Knorr and Pfirsch [**1**] on the exponential property for analytically smooth frequency and outline an elementary proof. An elementary proof of the transcendental property [**1**], [**2**], [**3**], [**4**] falls out as a bonus.

The method rests on an asymptotic split of the angle variable into its algebraic and exponential parts and on a Fourier representation of the action change with respect to an intrinsic pendulum time.

A precise order-estimate for the action change is reported for a class of frequency functions of practical interest (Theorem 2).

2. **Main results.** Let $q(t;\varepsilon)$ be a solution of $d^2q/dt^2+\omega^2q=0$ with $\omega=\omega(\tau)$, $\tau=\varepsilon t$, $0<\varepsilon<1$, and let $2P(t)=\omega q^2+\omega^{-1}(dq/dt)^2$. The following hypotheses will be distinguished.

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