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BEST APPROXIMATION BY COMPACT OPERATORS. II

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Introduction. Let H be an infinite-dimensional complex Hilbert space, and let $\mathscr{B}(H)$ (resp. $\mathscr{C}(H)$) be the algebra of all bounded (resp. compact) linear operators on H. It is well known **[4]**, **[6]** that $\mathscr{C}(H)$ is proximinal in $\mathscr{B}(H)$, that is, every $T \in \mathscr{B}(H)$ has a best approximation from the subspace $\mathscr{C}(H)$. Indeed, it has recently been noted by Fakhoury **[2]** that there is a continuous selection for the associated metric projector. We denote the metric complement of $\mathscr{C}(H)$ in $\mathscr{B}(H)$ by $\mathscr{C}(H)^0$; by definition, $\mathscr{C}(H)^0=\{T \in \mathscr{B}(H): \|T\|=\operatorname{dist}(T, \mathscr{C}(H))\} \equiv \{T \in \mathscr{B}(H): 0$ is a best compact approximation to $T\}$. Such operators (other than 0) are at maximum possible distance from $\mathscr{C}(H)$; they may thus be called "anti-compact" (called extremally noncompact by Coburn **[1]**). Every operator T is the sum of a compact operator and an anti-compact operator; we can give one such splitting explicitly in terms of the polar decomposition of T. It is the purpose of this note to announce a series of results on the contents and structure of $\mathscr{C}(H)^0$, with particular reference to the proper subset consisting of those operators for which 0 is the unique best compact approximation that commutes with the given operator. Full details and proofs of the theorems will appear elsewhere.

1. **Failure of unicity.** The seminal result that motivates and directs our subsequent investigation is the rather striking failure, in all cases, of uniqueness of best compact approximations.

THEOREM 1. *Let T be any noncompact operator in $\mathscr{B}(H)$. Then the set of all best compact approximations to T is a (closed, bounded) convex set of infinite dimension.*

It is clear that the metric complement of a Chebyshev subspace of a Banach space X is a (closed) nowhere dense subset of X, and simple examples show that the metric complement of a non-Chebyshev subspace

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