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THE HADAMARD THREE-CIRCLES THEOREMS FOR PARTIAL DIFFERENTIAL EQUATIONS

BY R. VÝBORNÝ[1]

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1. The famous Hadamard three-circles theorem of the complex function theory has been generalized to solutions of elliptic and parabolic equations. For references as well as for some interesting applications we refer to [3]. The purpose of this note is to show that (a) three circles (spheres)-theorems lead naturally to a sharpened version of the boundary point maximum principle (see [1], [2]), and (b) to prove a Hadamard type theorem for a quasilinear equation.

2. Let $G=\{x; x \in \boldsymbol{R}^n, |x|<a\}$, u be a nonconstant solution of $\Delta u \geqq 0$, which is of class C^2 in G and continuous in $\bar{G}$. Let

$$M(r) = \operatorname{Max}\{u(x): |x| = r\} \tag{1}$$

for $0<r\leqq a$. The strong maximum principle implies that $M(r)$ is a strictly increasing function of r. The Hadamard theorem states that $M(r)$ is a convex function of s, where $s=\log r$ for $n=2$ and $s=-r^{2-n}$ for $n>2$. Define $f(s)=M(r)$. Since f is a convex function it possesses a left-hand derivative $f'_-(s)$ on $(0, s(a)]$, and since s is a differentiable function of r M has the left-hand derivative $M'_-(r)=f'_-(s)(ds/dr)$. (Note that although the chain rule is not generally valid for one-sided derivatives it can be used here. Note also that $f'_-(a)$ can be infinite.) It will be proved now that $M'_-(a)>0$. Assume contrary to what one wishes to prove that $M'_-(a)\leqq 0$. Then we would have $f'_-(s)\leqq 0$ at $s=s(a)$ and hence for all $s \in (0, s(a)]$ since f'_- is increasing. Hence $M'_-(r)\leqq 0$ for $r \in (0, a]$. However $M'_-(r)\geqq 0$ since M is increasing; and it would follow that $M'_-(r)=0$ and $M(r)=$const. By the strong maximum principle u would be constant—contradiction. Hence we have proved $M'_-(a)>0$.

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