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THE ITERATIVE SOLUTION OF THE EQUATION $y \in x + Tx$ FOR A MONOTONE OPERATOR T IN HILBERT SPACE[1]

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ABSTRACT. Suppose T is a multivalued monotone operator with open domain $D(T)$ in a Hilbert space and $y \in R(I + T)$. Then there exist a neighborhood $N \subset D(T)$ of $\bar{x} = (I + T)^{-1}y$ and a real number $\sigma_1 > 0$ such that for any $\sigma \geqq \sigma_1$, any initial guess $x_1 \in N$, and any single-valued section T_0 of T, the sequence generated from x_1 by

$$x_{n+1} = x_n - (n + \sigma)^{-1}(x_n + T_0x_n - y)$$

remains in $D(T)$ and converges to $\bar{x}$ with estimate $\|x_n - \bar{x}\| = O(n^{-1/2})$. The sequence $\{x_n + T_0x_n\}$ converges $(C, 1)$ to y. No continuity assumptions of any kind are imposed on T_0.

If H is a real or complex Hilbert space with inner product $(\cdot, \cdot)$, a *multivalued monotone operator* on H is a subset T of $H \times H$ for which $\operatorname{Re}(u - v, x - y) \geqq 0$ whenever $[x, u]$, $[y, v] \in T$. We write Tx for $\{y \in H : [x, y] \in T\}$, $D(T) = \{x : Tx \neq \varnothing\}$ (the *effective domain* of T), $T(A) = \bigcup \{Tx : x \in A\}$ if $A \subset H$, and $R(T) = T(H)$. I denotes the identity operator on H, so $I + T = \{[x, x + y] : [x, y] \in T\}$ and $(I + T)^{-1} = \{[x + y, x] : [x, y] \in T\}$. T is *locally bounded* at x if there exists a neighborhood N of x (in the norm topology) for which $T(N)$ is bounded. A single-valued section of T is a subset T_0 of T for which T_0x is a singleton set for each x in $D(T)$; we follow the traditional abuse of terminology and refer to T_0x as the *element* in the singleton set.

One of the earliest problems in the theory of monotone operators was to solve the equation $y \in x + Tx$ for x, given an element y of H and a monotone operator T. The initial existence theorems (Vaĭnberg [**10**], Zarantonello [**12**]) were constructive in nature, but assumed that the operator T was single-valued and Lipschitzian; later existence results (Minty [**7**], Browder [**1**]) were proven under unusually weak continuity assumptions on T, but were nonconstructive in nature. Subsequent iteration methods have weakened the Lipschitz assumptions on T (Petryshn [**8**], Zarantonello [**11**]).

In this note we return to the iterative techniques, with the difference that we make *no* assumptions of continuity. Supposing that the equation $y \in x + Tx$ has a solution x, we calculate that solution as the limit of an iteratively constructed sequence with an explicit error estimate. Naturally,

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