BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 6, November 1973

THE AFFINE STRUCTURES ON THE REAL TWO-TORUS. I

BY T. NAGANO[1] AND K. YAGI

Communicated by S. S. Chern, April 17, 1973

We wish to complete the study of the affine structures on the real affine 2-tori T^2, following N. H. Kuiper [2], J. P. Benzecri [1] and others. The category of the affine manifolds is defined, as usual, by the manifolds equipped with maximal atlas whose coordinate transformations are affine transformations $y^i = \Sigma_j a^i_j x^j + b^i$, a^i_j, $b^i \in R$, in the cartesian space R^n, and by the maps which are expressed locally with affine transformations in terms of the affine charts.

Our main result asserts that the affine structures on T^2 are completely determined by the holonomy groups, in which, however, the concept of the holonomy group requires a slight modification as follows.

Given an affine manifold M, its universal covering manifold $M^\sim$ with the induced affine structure is immersed equidimensionally into R^n by an affine map d. The map d gives rise to a homomorphism $\eta: \pi_1(M) \to A(R^n)$ of the fundamental group into the affine group $A(R^n)$ in such a way that d is $\pi_1(M)$-equivariant with respect to the action of $\pi_1(M)$ on R^n through η. The image of η is called the holonomy group H of M, which is unique up to an inner automorphism of $A(R^n)$. Here $A(M)$, in general, denotes the affine automorphism group of the affine manifold M. When the image $dM^\sim$ is not simply connected, we switch to its universal covering $(dM^\sim)^\sim$ from R^n; that is, we construct an affine immersion: $d^*: M^\sim \to (dM^\sim)^\sim$ which covers d and a homomorphism $\eta^*: \pi_1 M \to A((dM^\sim)^\sim)$ accordingly. Now the modified holonomy group H^* of M is by definition the image $\eta^*(\pi_1 M)$. When $dM^\sim$ is simply connected, we simply put $H^* = H$. At any rate H^* can be regarded as a subgroup of the universal covering group $A(R^2)^\sim$ of $A(R^2)$.

THEOREM 1. *Two affine structures on T^2 are isomorphic if and only if the modified holonomy groups are conjugate in $A(R^2)^\sim$.*

The difficulty in the proof lies in establishing that d is a covering map onto $dM^\sim$. The difficulty may be illustrated by the fact that a surjective immersion of R^2 onto itself is not always a diffeomorphism. In any case, that d is a covering implies that T^2 is affine isomorphic with $(dM^\sim)^\sim/H^*$. In order to describe the classification of H^* it is convenient to state the following theorem.

AMS(MOS) subject classifications (1970). Primary 53C05, 57D15.
[1] Partially supported by NSF GP-29662.