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EXAMPLES IN THE THEORY OF THE SCHUR GROUP

BY CHARLES FORD AND GERALD JANUSZ

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Let K be a subfield of a cyclotomic extension of the rational field Q. The Schur group of K is the subgroup $S(K)$ of the Brauer group of K consisting of those classes of central simple K algebras represented by an algebra which appears as a direct summand of a group algebra $Q[G]$ for some finite group G. For a prime p let $S(K)_p$ denote the subgroup consisting of elements having p-power order. It is known by [1] that $S(K)_p$ can have an element of order p^a only when a primitive p^a root of unity, ε_{p^a}, is in K.

Suppose K is a field which satisfies $Q(\varepsilon_{p^a}) \subseteq K \subseteq Q(\varepsilon_n)$ and p^a is the highest power of p dividing n. It is known that

$$S(K)_p = K \otimes S(Q(\varepsilon_{p^a}))_p \tag{1}$$

in the case $K = Q(\varepsilon_n)$. That is every element in $S(K)_p$ is represented by an algebra $K \otimes B$ with B central simple over $Q(\varepsilon_{p^a})$ [2].

The assertion (1) also holds for K if p does not divide $(Q(\varepsilon_n): K)$. In this paper we present, for each prime p, fields K for which (1) does not hold.

Let p be a prime and r and s distinct primes such that $r \equiv s \equiv 1 \bmod p$. Then the field $L = Q(\varepsilon_p, \varepsilon_r, \varepsilon_s)$ has two nontrivial automorphisms σ, τ which satisfy

(i) $\sigma^p = \tau^p = 1$

(ii) σ fixes ε_p and ε_r; τ fixes ε_p and ε_s.

Let K be the subfield of L fixed by $\langle \sigma, \tau \rangle$. Let A be the algebra defined by

$$A = \sum L u_\sigma^i u_\tau^j;$$

$$u_\sigma^p = u_\tau^p = 1, \qquad u_\sigma u_\tau = \varepsilon_p u_\tau u_\sigma;$$

$$u_\sigma x = \sigma(x) u_\sigma, \qquad u_\tau x = \tau(x) u_\tau \quad \text{for } x \text{ in } L.$$

Then A is central simple over K and is a simple component of the group algebra $Q[G]$ where G is the group of order p^3rs generated by u_σ, u_τ, ε_{prs}. We use this algebra for several examples.

Let f_r be the exponent of $r \bmod s$; that is, f_r is the least positive integer f such that $r^f \equiv 1 \bmod s$. Similarly let f_s be the exponent of $s \bmod r$.

THEOREM. (1) *If $p \mid f_r$ then the r-local index of A is p. In particular A has index p if either $p \mid f_r$ or $p \mid f_s$.*

(2) *If A has r-local index p and p^2 divides either $r - 1$ or f_r then A is not*

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