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A FATOU THEOREM FOR THE GENERAL ONE-DIMENSIONAL PARABOLIC EQUATION[1]

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1. **Introduction.** Let R be a finite or infinite one-dimensional open interval. Our main purpose here is to characterize all positive weak solutions of the equation

$$\partial u/\partial t = \mathscr{D}u = a(x)u_{xx} + b(x)u_x + c(x)u \quad \text{in } R \times (0, T) \tag{1.1}$$

where $T \leqq \infty$. Here $a(x) \geqq 0$, $c(x) \leqq M$,[2] $1/a(x)$, $b(x)/a(x)$ and $c(x)/a(x)$ are locally integrable in R, but otherwise the coefficients are unrestricted. The results below extend characterizations of Widder ([**17**], [**18**]) for positive solutions of the heat equation (see §1.1). In particular, we find that all positive solutions of (1.1) are of the form

$$u(x, t) = \int_R p(t, x, y)F(dy), \tag{1.2}$$

where $p(t, x, y)$ is the fundamental solution of (1.1), if and only if the Green's function of (1.1) is not of trace class at either endpoint of R. While these results are one-dimensional, they do have the advantage that they are complete, and suggest possible generalizations in higher dimensions. Proofs will appear elsewhere.

Equation (1.1) can always be transformed into a similar equation with $c(x) = 0$; assume for the moment $c = 0$ in (1.1). Then by a simple change of variables we can write (1.1) in "Feller form"

$$\partial u/\partial t = \mathscr{D}u = (d/dm(x))(du/ds(x)) \quad \text{in } R \times (0, T) \tag{1.3}$$

where $m(x)$ and $s(x)$ are increasing; we can also consider equation (1.3) for an arbitrary strictly increasing continuous function $s(x)$ (a "scale") and Borel measure $m(dx)$ which is positive on open sets in R. Then $\mathscr{D}$ becomes

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[2] The condition $c(x) \leqq M$ can be removed if there exists some $g(x) > 0$ in R such that $\mathscr{D}g = \lambda g$ for some λ.