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A GLOBAL THEORY OF STEADY VORTEX RINGS IN AN IDEAL FLUID

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The question of whether the equations governing the motion of an inviscid, incompressible fluid admit solutions representing steady vortex rings has not been studied widely, despite the central place of such rings in the theory of vortex motion initiated by Helmholtz [**1**] in 1858. Hill [**2**] discovered in 1894 an explicit particular solution for which the 'ring' is actually a ball in $\boldsymbol{R}^3$. More recently, there have appeared local existence proofs for (a) steady rings of small cross-section [**3**], [**4**], [**5**] and (b) steady rings close to Hill's vortex, but homeomorphic to a solid torus [**6**]; these two cases represent opposite extremes. The present note outlines what we believe to be the first global existence theory for steady vortex rings.

By a steady vortex ring we mean a figure of revolution $\mathscr{A}$ that is expected to be homeomorphic to a solid torus in most cases, and is associated with a continuous, axi-symmetric, solenoidal vector field $\boldsymbol{q}$ (the fluid velocity) defined in a cylinder V or in the whole of $\boldsymbol{R}^3$, and having the following properties when we take axes fixed in the ring $\mathscr{A}$. (a) Both $\mathscr{A}$ and $\boldsymbol{q}$ do not vary with time; (b) the vorticity $\boldsymbol{\omega} \equiv \operatorname{curl} \boldsymbol{q}$ has positive magnitude in $\mathscr{A}$, vanishes in $V - \mathscr{A}$ or $\boldsymbol{R}^3 - \mathscr{A}$, and satisfies a nonlinear equation of motion which, among other things, determines the boundary of $\mathscr{A}$; (c) $\boldsymbol{q}$ has a prescribed normal component on ∂V, or tends to a constant value at infinity in $\boldsymbol{R}^3$. For the case of a cylinder V these requirements lead to equations (1) and (2) below, and we solve the resulting problem by means of the calculus of variations in the large [**7**], [**8**]. We also use Steiner symmetrization [**9**], the generalized maximum principle [**10**] and certain a priori estimates to describe the solution and to deal with two limiting cases: (i) that when the nonlinear term in the differential equation (2a) is discontinuous, and (ii) that when the domain of $\boldsymbol{q}$ is the whole of $\boldsymbol{R}^3$ (so that the usual compactness theorems, needed for the solution of variational problems, do not hold).

1. **Preliminaries.** Let $X = [X_1, X_2, X_3] = [r\cos\theta,\ r\sin\theta,\ z]$ be a point of $\boldsymbol{R}^3$, so that (r, θ, z) are cylindrical coordinates. Consider the axi-symmetric flow of an inviscid fluid, of uniform density ρ, in a cylinder V that is here represented in a meridional plane ($\theta =$ const) by the domain D;

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