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TRANSFORMATIONS BY FUNCTIONS IN SOBOLEV SPACES AND LOWER SEMICONTINUITY FOR PARAMETRIC VARIATIONAL PROBLEMS

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Introduction. In this work we are concerned with integrals of the form

$$\int_{\Omega} f(x, \varphi(x), \varphi_x(x))\, d\mathscr{L}_m(x) \tag{0.1}$$

where $\Omega \subset R_m$ is a bounded domain, $\mathscr{L}_m$ denotes m-dimensional Lebesgue measure, $\varphi = (\varphi^1, \ldots, \varphi^n)^{\mathrm{tr}}$ is a continuous transformation from Ω to R_n ($m \leqq n$) belonging to some Sobolev space $W_p^{1,\mathrm{loc}}(\Omega)$, and φ_x is the (almost everywhere defined) matrix function $(\partial\varphi^i/\partial x_j)$. Here f is a real-valued function on $\Omega \times R_n \times M_{n \times m}$, where $M_{n \times m}$ is the space of $n \times m$ matrices. Such integrals arise in the analysis of various variational problems. The success of the analysis, by direct methods, of *nonparametric* variational problems, where it is appropriate to make direct convexity hypotheses on $f(x, z, \cdot)$, is well known. In contrast, the study of *parametric* variational problems by these methods has been less successful. In this latter class of problems, which includes Plateau's problem, the fact that the integral has special behavior under transformation of independent variables implies that the integrand cannot possess the strong type of convexity properties referred to above. Only for the case $m = 2$ has there been available a lower semicontinuity result for such problems, in what would seem to be their natural context, namely with φ continuous and belonging to $W_m^1(\Omega)^n$ and with sequences $\{\varphi^k\}$ converging uniformly to φ. An important example due to Besicovitch [**1**] has indicated that the situation here is a very delicate one. For $m > 2$, a result of this type was obtained by Morrey [**7**, Theorem 9.2.1] under the additional assumption that φ is locally Lipschitz in Ω.

The present note provides, under natural hypotheses for parametric integrands [**7**, p. 356], a lower semicontinuity result for the general case of $\varphi \in W_m^1(\Omega)^n \cap C(\Omega)^n$, $m \geqq 2$. The result is obtained by techniques due to McShane [**5**] and Morrey [**7**], combined with a measure-geometric estimate for mappings $\varphi \in W_p^1(\Omega)^n \cap C(\Omega)^n$, $p > m$. This estimate, together with a few additional results concerning mappings φ of this type, is presented in the first part of this note.

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Key words and phrases. Change of variables formula, Sard lemma, Sobolev spaces, parametric variational problem, lower semicontinuity, Lebesgue-Fréchet surface area.

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