BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

TRANSFORMATION GROUPS AND BIFURCATION AT MULTIPLE EIGENVALUES[1]

BY D. H. SATTINGER

Communicated by Hans F. Weinberger, January 12, 1973

1. Topological degree arguments show that bifurcation must take place at eigenvalues of odd multiplicity, while examples show bifurcation may not take place at eigenvalues of even multiplicity. The general problem of bifurcation at multiple eigenvalues is one which does not readily submit to a complete solution, so the approach must proceed by special cases. One important class for which a more detailed analysis is possible is that of problems invariant under a transformation group. The purpose of this note is to present some results in this direction. We would also like to call the reader's attention to the recent interesting work of D. Ruelle.

2. Consider a nonlinear problem

$$(L_0 + \lambda L_1)u + N(u) = 0 \tag{1}$$

where L_0, L_1 and N are continuous operators from a Banach space X to $Y(X \subset Y)$. We assume that N is Fréchet differentiable, that $N'(0) = 0$, and that $L_0 + \lambda L_1$ is a Fredholm operator for all λ. (This means that the dimension of the null space of $L_0 + \lambda L_1$ is equal to the codimension of its range in Y, and that its range is closed in Y.) Let G be the group generated by S^1 (the unit circle with the usual addition) together with an inversion operator $i(i^2 = e$, where e is the identity operator). Let T_g be a representation of G on Y. We label elements of S^1 by γ, δ, where $0 \leqq \gamma$, $\delta < 2\pi$. Then $T_\delta T_\gamma = T_{\delta+\gamma}$. We assume that $L_0 + \lambda L_1$ and N commute with T_g for all g in G.

Let $\mathcal{N}_\lambda$ be the null space of $(L_0 + \lambda L_1)$ and let $\{\lambda_j\}$ be the real numbers λ for which $\mathcal{N}_\lambda$ is nontrivial. $\mathcal{N}_{\lambda_j}$ is invariant under G, and we assume $\mathcal{N}_{\lambda_j}$ is finite dimensional. If $\mathcal{N}_{\lambda_j}$ is irreducible, it is either one or two dimensional. Since bifurcation at simple eigenvalues is well understood (see [**5**] or [**3**]), let us consider the case $\dim \mathcal{N}_{\lambda_j} = 2$. The eigenvalues of T_i are ± 1, so in $\mathcal{N}_{\lambda_j}$ there is one vector φ_j such that $T_i\varphi_j = \varphi_j$. Restricting (1) to the subspaces X' and Y' of all vectors u such that $u = T_i u$, the corresponding null space $\mathcal{N}_{\lambda_j}^+$ is one dimensional. (Since N commutes with T_i, $T_i N(u) = N(T_i u) = N(u)$ if u is in X^+.) We can now apply the usual bifurcation arguments to obtain a bifurcating curve of solutions

---

AMS (MOS) subject classifications (1970). Primary 35J60, 20C35.
Key words and phrases. Transformation groups, bifurcation theory.
[1]Partially supported by NSF GP-27973.