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DERIVATION RANGES AND THE IDENTITY

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Introduction. If $\mathfrak{A}$ is a C^*-algebra containing the identity and T belongs to $\mathfrak{A}$, then the (inner) derivation induced by T is the operator Δ_T acting on $\mathfrak{A}$ which maps X (in $\mathfrak{A}$) to $TX - XT = [T, X]$. It has been known for many years that I (the identity in $\mathfrak{A}$) is not in the range of Δ_T for any T (I is not a commutator)[4]. J. P. Williams has asked [5] if there is a T in $\mathfrak{A}$ such that I is in the uniform closure of the range of Δ_T. In this paper we show that such T's do exist. In fact, if $\mathscr{A}(\mathfrak{A}) = \{T \in \mathfrak{A} : I \text{ is in the closure of the range of } \Delta_T\}$ we will show that there is a C^*-algebra $\mathfrak{A}$ such that $\mathscr{A}(\mathfrak{A})$ is uniformly dense in $\mathfrak{A}$. It then follows that $\mathscr{A}(\mathscr{B}(\mathscr{H}))$ is nonempty where $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded linear operators acting on complex infinite dimensional Hilbert space.

Ampliations. In what follows $\mathscr{H}$ will always denote a *separable* infinite dimensional complex Hilbert space. Given T in $B(\mathscr{H})$ the *ampliation* of T is denoted by $T \otimes I$ and is by definition the operator acting on the direct sum of $\aleph_0$ copies of $\mathscr{H}$ which is determined by the matrix

$$T \otimes I = \begin{bmatrix} T & 0 & 0 & \\ 0 & T & 0 & \\ 0 & 0 & T & \\ & & & \cdot \end{bmatrix}.$$

It is well known that the commutant of $T \otimes I$ contains the set of matrices of the form

$$I \otimes S = \begin{bmatrix} s_{11}I & s_{12}I & s_{13}I & \\ s_{21}I & s_{22}I & s_{23}I & \\ s_{31}I & s_{32}I & s_{33}I & \\ & & & \cdot \end{bmatrix},$$

where $S = (s_{ij})$ is the matrix of some element of $\mathscr{B}(\mathscr{H})$. It is also well known that $I \otimes S$ is unitarily equivalent to $S \otimes I$.

In [1] Brown and Pearcy showed that there are sequences of operators X_n and Y_n in $\mathscr{B}(\mathscr{H})$ such that $[X_n, Y_n]$ tends uniformly to the identity as

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