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ALMOST NORMAL OPERATORS, THEIR SPECTRA AND INVARIANT SUBSPACES[1]

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1. **Introduction.** This lecture will deal with a few classes of operators which in some sense are close to being normal. For the sake of definiteness the underlying Hilbert space H will be taken to be infinite dimensional and separable. Occasionally, it may be convenient to let H be finite dimensional, but, in this case, the operators considered will usually reduce to normal ones. By an operator T is meant a bounded operator, so that T is linear, is defined on the whole space H, and satisfies $\|Tx\| \leqq \text{const}\|x\|$ for all x in H. By the spectrum, $\text{sp}(T)$, of T is meant the set of complex numbers z for which $(T - zI)^{-1}$ fails to exist as a bounded operator. Recall that T is said to be normal if $T^*T = TT^*$.

It seems not out of place here to remark that the "almost" of the title is a bona fide adverb and not part of some compound adjective describing a new kind of operator. In line with this remark it is noteworthy that many classes of almost normal operators which have received attention can be produced by supplying an appropriate prefix or adverb to the root adjective "normal." A few of these prefixes are quasi, sub, hypo, semi, (G_1) (which appears to be a ringer in this group), para (cf. [**11**], [**19**]), some adverbs: "nearly," "vaguely" and others. Suffixes apparently have not caught on as well, although "oid" is popular with various root adjectives preceding it, witness normaloid, convexoid and spectraloid; cf. Halmos [**16**, p. 114]. For classifications of some almost normal operators, see, e.g., Furuta [**11**, p. 595], Gustafson [**13**, p. 37], Stampfli [**36**, p. 473].

The classes of operators we shall consider here can be arranged as follows, each inclusion being proper:

$$\text{normal} \subset \text{quasinormal} \subset \text{subnormal} \subset \text{hyponormal} \subset \text{seminormal} \subset (G_1).$$

These will be discussed briefly, with particular consideration to necessary and/or sufficient conditions on the spectrum assuring that the operator is (or is not) normal or has (or does not have) a normal part. Finally, in

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