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A FIXED POINT THEOREM FOR MAPPINGS IN SCALED METRIC SPACES, WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. A fixed point theorem is given for mappings of scaled metric spaces. This theorem applies to the Cauchy problem for partial differential equations, to singular differential equations, and to the Goursat problem.

1. **Introduction.** In this note we give a fixed point theorem for a class of mappings of a space with a two-parameter scale of metrics. This theorem contains as special cases the classical Cauchy-Kovalevska theorem (see Rosenbloom [**6**], [**7**]), as well as the generalizations of Ovsjannikov [**3**], Treves [**9**], and Duchateau and Treves [**1**]. It can also be applied to some classes of partial differential equations such as those treated by Rosenbloom [**8**], to the Goursat problem, and to other cases of the Riquier problem (see [**4**], [**5**]).

We shall say that $\{d_{r,s}|r, s \in (0, 1]\}$ is a *scale of metrics* on the space S if each $d_{r,s}$ is a metric on S, and $d_{r,s}$ is nondecreasing in each of the variables r and s. If A is a subset of S, then we denote by $A_{r,s}$ the completion of A with respect to $d_{r,s}$. For $\alpha > 0$, we shall denote by $\delta^{\alpha}_{r,s}$ the metric

$$\delta^{\alpha}_{r,s}(u, v) = r^{\alpha} \sup_{0<\rho\leqq r} \rho^{-\alpha} d_{\rho,s}(u, v).$$

Let c be a function on $R^+ = \{x|x > 0\}$ to R^+, let $\mu, \nu \in R^+$. We shall say that T is a mapping of class $K(c, \mu, \nu)$ if, for $0 < r \leqq 1, 0 < s < \sigma \leqq 1$, T is a mapping of $S_{r,\sigma}$ into $S_{r,s}$ and satisfies

$$d_{r,s}(Tu, Tv) \leqq \frac{c(n)r^{\mu}}{c(n + \mu)(\sigma - s)^{\nu}} \delta^{n}_{r,\sigma}(u, v) \tag{1}$$

for $u, v \in S, n > 0$. For any $\lambda > 0$ we denote by Δ_{λ} the metric

$$\Delta_{\lambda} = \sup\{d_{r,s}|r^{\mu} \leqq \lambda(1 - s)^{\nu}\}$$

and by $A(\lambda)$ the completion of the set A with respect to Δ_{λ}.

THEOREM. *Suppose that* $\{d_{r,s}\}$ *is a scale of metrics on S and T is a mapping of class* $K(c, \mu, \nu)$, *and that the power series*

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