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# EXTREMAL PLANE QUASICONFORMAL MAPPINGS WITH GIVEN BOUNDARY VALUES

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1. **Introduction.** Let $\Omega_i$, $i = 1, 2$, be regions in the complex plane, $f(z)$ a quasiconformal mapping of $\Omega_1$ onto $\Omega_2$. Let $Q_f$ denote the class of all quasiconformal mappings of $\Omega_1$ onto $\Omega_2$ which have the same boundary values as $f$. A mapping $f^* \in Q_f$ will be called *extremal* for its boundary values if it is $K^*$-quasiconformal and if there exists no $K$-quasiconformal mapping in $Q_f$ with $K < K^*$. The quantity $K^* = K^*(f)$ is the extremal dilatation for the class $Q_f$. (As is well known [3], there may be more than one $K^*$-quasiconformal mapping in the class $Q_f$.) In the present account, which is only an abstract, we restrict ourselves to the case $\Omega_1 = \Omega_2 = E = \{|z| < 1\}$. Generalizations to Riemann surfaces will be referred to in a detailed account, giving proofs, further results, and applications that is to appear elsewhere.

In what follows, the $L^1$ norm $\int\int_E |\varphi(z)|\,dx\,dy$ of functions $\varphi(z)$ *holomorphic* in $E$ will be denoted by $\|\varphi\|$.

In 1969, R. S. Hamilton [1] proved the following: *If $f^* \in Q_f$ is an extremal mapping, $\kappa^*(z) = f^*_{\bar z}/f^*_z$, then*

$$\sup_{\|\varphi\| \leqq 1} \left| \iint_E \kappa^*(z)\varphi(z)\,dx\,dy \right| = k^*(f) = \frac{K^*(f) - 1}{K^*(f) + 1}. \tag{1.1}$$

A central result of the present work is (§3) that condition (1.1) *characterizes* extremal mappings of $E$.

2. **Estimates for $K^*(f)$.** The following is a generalization of an inequality proved in [2] from the case $K^*(f) = 1$ to arbitrary $K^*(f)$.

THEOREM 2.1. *If $f(z)$ is a quasiconformal self-mapping of $E$, $\kappa(z) = f_{\bar z}/f_z$, and if $\varphi(z)$ is holomorphic in $E$, then*

$$\left| \iint_E \frac{\kappa(z)}{1-|\kappa(z)|^2}\varphi(z)\,dx\,dy \right| \leqq \frac{k^*(f)}{1+k^*(f)}\|\varphi\| + \iint_E \frac{|\kappa(z)|^2}{1-|\kappa(z)|^2}|\varphi(z)|\,dx\,dy. \tag{2.1}$$

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[1] Supported by GP 27275.

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