PROJECTIVE MODULES FOR FINITE CHEVALLEY GROUPS

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1. **Introduction.** The irreducible modular representations of the finite Chevalley groups (and their twisted analogues) have been described by C. W. Curtis and R. Steinberg (see [1], [2], [8]). In this note we outline some parallel results on principal indecomposable modules (PIM's), for which proofs will appear elsewhere. The groups $SL(2, q)$ are already treated in detail in [5], based in part on the method of A.V. Jeyakumar [6].

K denotes an algebraically closed field of prime characteristic p, over which all modules are assumed to be finite dimensional. Our notation resembles that of [4], [5]: G is a simply connected algebraic group (of simple type), $\mathfrak{g}$ its Lie algebra, $\mathscr{U}$ the restricted universal enveloping algebra of $\mathfrak{g}$, G_q the group of rational points of G over a field of q elements, $\mathscr{R}_q$ the group algebra of G_q over K. (When $q = p$, we write simply $G, \mathscr{R}$.)

EXAMPLE. $G = SL(2, K)$, $\mathfrak{g} = \mathfrak{sl}(2, K)$, $G_q = SL(2, q)$.

The set Λ of restricted highest weights (determined by integers between 0 and $p - 1$) indexes the (classes of) irreducible modules M_λ for $\mathscr{U}$ (or $\mathscr{R}$). If $\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_k p^k$ ($\lambda_i \in \Lambda$), then the twisted tensor product modules $M_\lambda = M_{\lambda_0} \otimes M_{\lambda_1}^{(p)} \otimes \cdots \otimes M_{\lambda_k}^{(p^k)}$ exhaust the (classes of) irreducible modules for $\mathscr{R}_q$ ($q = p^{k+1}$) and for G (as k runs over all nonnegative integers). Denote by $U_\lambda, R_\lambda, R_\lambda$ the respective PIM of $\mathscr{U}, \mathscr{R}, \mathscr{R}_q$ having top composition factor $M_\lambda, M_\lambda, M_\lambda$. The only irreducible module which is also projective is the Steinberg module $M_\sigma = U_\sigma = R_\sigma$, $\sigma = (p - 1)\delta$, δ = half-sum of positive roots. A similar statement is true for $M_\sigma = R_\sigma$ ($\boldsymbol{\sigma} = \sigma + \sigma p + \cdots + \sigma p^k$).

2. **Projective modules.**

LEMMA. *Let V, W be modules for the restricted universal enveloping algebra of a restricted Lie algebra, with W projective. Then $V \otimes W$ is also projective.*

This is proved in [7]. The analogous statement for the group algebra of a finite group is well known [3, Exercise 2, p. 426].

We apply the lemma as follows. For $\mu \in \Lambda$, define $T_\mu = M_\mu \otimes M_\sigma$ (σ as above). This is a module for $G, \mathscr{R}, \mathscr{U}$, and is projective for $\mathscr{R}, \mathscr{U}$ (since M_σ is). In particular, T_μ is the direct sum of certain $\mathscr{U}$-modules U_λ.