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WAVE FRONT SETS AND THE VISCOSITY METHOD

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Consider the initial value problem

$$\frac{\partial}{\partial t} u + \sum_{j=1}^{n} A_j(x,t) \frac{\partial}{\partial x_j} u = 0, \qquad u(x,0) = \Phi(x). \tag{1}$$

Here $A_j(x, t)$ are smooth $k \times k$ matrix-valued functions on $R^n \times R$, constant for large $|x| + |t|$, and $u_0 \in H^s(R^n)$ for some $s \in R$. We assume that (1) is strictly hyperbolic in the sense that $\sum_1^n \xi_j A_j$ has real and distinct eigenvalues for all $\xi = (\xi_1, \ldots, \xi_n) \neq 0$. The solution to this problem is a weak, i.e., distribution, limit of solutions u_ε of the problem

$$\frac{\partial}{\partial t} u_\varepsilon + \sum_{j=1}^{n} A_j(x,t) \frac{\partial}{\partial x_j} u_\varepsilon = \varepsilon B(x, D_x) u_\varepsilon, \qquad u_\varepsilon(x,0) = \Phi(x), \tag{2}$$

where $\varepsilon > 0$ and $B(x, D_x)$ is an elliptic operator of order two, whose principal symbol is a $k \times k$ matrix with positive eigenvalues[2]. In fact, this is the well-known technique of parabolic regularization, or the "viscosity" method.

Our interest in these problems arises from the study of systems of nonlinear conservation laws. In studying computer output of such systems, we observed that for certain viscosity matrices, suggested in [**1**], the solutions u_ε were not converging to the correct solution, but were converging to a solution containing an extraneous shock wave. We discovered similar behavior for linear systems and it is this phenomenon which we investigate here.

It is known (see [**3**], [**5**]) that the singularities of the solution u to (1) propagate along bicharacteristics through the singularities of the initial data Φ. In our numerical experiments, for $n = 1$, $k = 2$, we took Φ to have a single singularity at the origin, and we knew that the singularity of the solution lay along precisely one bicharacteristic ray through the origin. However, we found that the approximate solutions seemed to be developing a singularity along the second bicharacteristic through the

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[2] We could more generally handle the case where the eigenvalues have positive real parts.