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REMARKS ON A PAPER OF D. LUDWIG

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We consider the reduced wave equation

$$\text{(A)} \qquad \Delta u + k^2 u = 0$$

with initial data prescribed on a hypersurface $Z \subset R^n$. It is well known that for points near Z, (A) has an asymptotic solution of the form:

$$\text{(B)} \qquad a_+(x, k)e^{ik\phi_+} + a_-(x, k)e^{ik\phi_-}$$

where $a_\pm(x, k)$ is an asymptotic series in k^{-1} and $\phi_\pm$ is a solution of the eikonal equation: $(\nabla\phi)^2 = 1$ with initial data $\phi_\pm = 0$ on Z. The nonlinear initial value problem defining $\phi_\pm$ may be solved locally by ray tracing, but in general these rays have an envelope (often called a caustic), so a global solution does not exist. This limits the domain in which an asymptotic series of the form (A) can be valid. In [**4**] Ludwig has derived an expansion valid near a caustic which involves the Airy function. In this note we show how Ludwig's result follows naturally from the combination of two well established mathematical disciplines: Maslov's global theory of asymptotics ([**5**], [**6**]) and the theory of singularities of differentiable mappings. In particular, the application of singularity theory leads to an asymptotic expansion in three dimensions at an intersection of the two focal surfaces, a question left unanswered by Ludwig.

The central idea in the Maslov theory is to consider multiple valued solutions of the eikonal equation. This is most elegantly done with the formalism of Lagrangian manifolds. (See for example [**3**].) If X is a smooth manifold of dimension n, a Lagrangian manifold is an n-dimensional submanifold of the cotangent bundle $T^*(X)$ on which the symplectic 2-form $\sum dx_i \wedge d\xi_i$ vanishes. For any function $\phi \in C^\infty(M)$, the graph of $d\phi$,

$$\Gamma(d\phi) = \{(x, d\phi_x) : x \in X\},$$

is a Lagrangian manifold Λ such that the natural projection $\pi : T^*(X) \to X$, restricted to Λ, is a diffeomorphism. Conversely, if Λ is a Lagrangian manifold and if $\pi : \Lambda \to X$ is a diffeomorphism, then locally Λ may be described as the graph of some function. Of course, in general $\pi|\Lambda$ will not be a diffeomorphism, but will have certain singularities. For example, a

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