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FUNCTIONS WITH A SPECTRAL GAP

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Introduction. In harmonic analysis, it is important to know how various properties of a function on $\boldsymbol{R}^n$ reflect themselves as restrictions on its *spectrum*, i.e., the support of its (distributional) Fourier transform. Thus, according to Paley and Wiener, a compact spectrum is characteristic of entire functions of exponential type. In this note we consider a milder restriction: it is only required of the spectrum that it be smaller than the whole n-space. Our results extend those of Levinson, Logan, Ehrenpreis and Malliavin; cf. also Boas [**1**]. Here we give only bare outlines of proofs; we employ standard vector notations: $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots x_n)$ are points of $\boldsymbol{R}^n$ and (t, x) denotes $\sum_1^n t_j x_j$; $|t| = (t, t)^{1/2}$, and dt denotes Haar measure on $\boldsymbol{R}^n$.

1. A *gap* in a distribution on $\boldsymbol{R}^n$ is a nonvoid open ball disjoint from its support. A *spectral gap* in a tempered distribution is a gap in its Fourier transform. In particular, an L^1 function f has a spectral gap if its Fourier transform $\hat{f}(x)$ vanishes on some nonvoid open set. Such f cannot decay too rapidly, by virtue of the following result of N. Levinson.

THEOREM A. *Let $f \in L^1(\boldsymbol{R})$, and suppose for some $\delta > 0$*

$$\int_0^\infty |f(t)|e^{\delta t}\, dt < \infty. \tag{1}$$

Then, if $\hat{f}(x)$ vanishes throughout any interval, it vanishes identically.

For the proof, one need only check [**4**, p. 74] that (1) implies that $\hat{f}(x)$ is the boundary value of a function holomorphic in a strip above the real axis. (Actually Levinson, *loc. cit.*, proves much deeper results, with (1) replaced by weaker hypotheses that do not force analyticity of $\hat{f}(x)$. An account of these, based on a new and simple method, will be given by me in a subsequent paper. The weaker Theorem A will serve as a basis for the present discussion.)

Theorem A admits a straightforward generalization to $\boldsymbol{R}^n$. Let us say that a convex cone K in $\boldsymbol{R}^n$ (all cones will be supposed to have vertex at the origin) is *minor* if there exists a unit vector $t^0 \in \boldsymbol{R}^n$ such that $\inf(t^0, t)$; $t \in K, |t| = 1$, is positive. Thus, a half-line in $\boldsymbol{R}^1$, or a sector of opening

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